Recognizing Interval Bigraphs by Forbidden Patterns

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Abstract

Let H be a connected bipartite graph with n nodes and m edges. We give an $O(n(m+n))$ time algorithm to decide whether H is an interval bigraph. The best known algorithm has time complexity $O(nm^6(m+n)\log n)$ and it was developed in 1997 [15]. Our approach is based on an ordering characterization of interval bigraphs introduced by Hell and Huang [11]. We transform the problem of finding the desired ordering to choosing strong components of a pair-digraph without creating conflicts. We make use of the structure of the pair-digraph as well as decomposition of bigraph H based on the special components of the pair-digraph. This way we make explicit what the difficult cases are and gain efficiency by isolating such situations. We believe our method can be used to find a desired ordering for other classes of graphs and digraphs having ordering characterization.

1 Introduction

A bigraph H is a bipartite graphs with a fixed bipartition into *black* and *white* vertices. (We sometimes denote these sets as B and W, and view the vertex set of H as partitioned into (B, W).) A bigraph H is called *interval bigraph* if there exists a family of intervals I_v, $v \in B \cup W$, such that, for all $x \in B$ and $y \in W$, the vertices x and y are adjacent in H if and only if I_x and I_y intersect. The family of intervals is called an *interval representation* of the bigraph H.

Interval bigraphs were introduced in [10] and have been studied in [5, 11, 15]. They are closely related to interval digraphs introduced by Sen et. al. [16], and in particular, our algorithm can be used to recognize interval digraphs (in time $O((m+n)n)$) as well.

Interval bigraphs generalize interval graphs. Interval graphs arise naturally in the process of modeling real-life situations, especially those involving time dependencies or other restrictions that are linear in nature. Therefore interval bigraph may be used to model real-life problems.

Recently interval bigraphs and interval digraphs became of interest in new areas such as graph homomorphisms, cf. [12].

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A bipartite graph whose complement is a circular arc graph, is called a *co-circular arc bigraph*. It was shown in [11] that the class of interval bigraphs is natural subclass of co-circular arc bigraphs, corresponding to those bigraphs whose complement is the intersection of a family of circular arcs no two of which cover the circle. There is a linear time algorithm for the recognition of co-circular arc bigraphs [14]. The class of interval bigraphs is a natural superclass of proper interval bigraphs (bipartite permutation graphs) for which there is a linear time recognition algorithm [11], [3].

Interval bigraphs can be recognized in polynomial time using the algorithm developed by H.Muller [15]. However, Muller's algorithm, [15], runs in time $O(nm^6(n+m)\log n)$. This is in sharp contrast with the recognition of *interval graphs*, for which several linear time algorithms are known, e.g., [2, 6, 7, 9, 13].

In [11, 15] the authors attempted to give a forbidden structure characterization of interval bigraphs, but fell short of the target. In this paper some light is shed on these attempts, as we clarify which situations are not covered by the existing forbidden structures. We believe our algorithm can be used as a tool for producing the interval bigraph obstructions. There are infinitely many obstructions and they are not fit into a few families of obstructions or at least we are not able to describe them in such a manner. However, the main purpose of this paper is devising an efficient algorithm for recognizing the interval bigraphs.

We use an ordering characterization of interval bigraphs introduced in [11]. Bigraph H is an interval bigraph if and only if its vertices admit a linear ordering $<$ without any of the forbidden patterns in Figure 1.

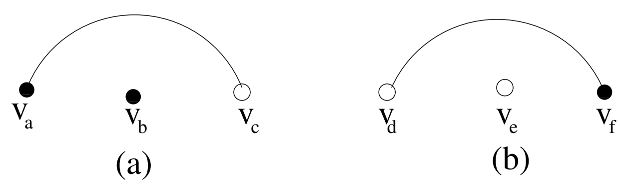

Figure 1: Forbidden Patterns

There are several graph classes that can be characterized by existence of ordering without forbidden pattern. One such an example is the class of interval graphs. A graph G is an interval graph if and only if there exists an ordering $<$ of the vertices of G such that none of the following patterns appears [12].

- $a < b < c$, $ac, bc \in E(G)$ and $ab \notin E(G)$
- $a < b < c$, $ac \in E(G)$ and $bc, ab \notin E(G)$

Proper interval graphs, co-comparability graphs, comparability graphs, chordal graphs, Convex bipartite graphs, co-circular arc bigraphs, proper interval bigraphs (bipartite permutation graph), interval bigraphs have ordering characterization without forbidden patterns.

The pair-digraph corresponding to the forbidden patterns in Figure 1 is the following digraph.

Let $H = (B, W)$ be a bigraph. The *pair-digraph* H^+ of H has pairs (vertices) (u, v), $u, v \in$

$V(H)$ with $u \neq v$. Note that there are two kinds of pairs in H^+ - pairs (u, v) where the vertices u, v have the same color, and pairs (u, v) where u, v have different colors.

- There is in H^+ an arc from (u, v) to (u', v) when u, v have the same color and $uu' \in E(H)$ and $vu' \notin E(H)$.
- There is in H^+ an arc from (u, v) to (u, v') when u, v have different colors and $vv' \in E(H)$ and $uv \notin E(G)$.

Note that if there is an arc from (u, v) to (u', v') then both $uv, u'v'$ are non-edges of H. The pair-digraph H^+ encodes the constraints in the ordering. To see that, suppose $<$ is an ordering of H without the forbidden patterns (see Figure 1). If $u < v$ and $(u, v)(u', v')$ is an arc in H^+, then $u' < v'$. For more details see the proof of Lemma 2.1. It follows from this definition that in H^+ there are only arcs between a pair with different colors and a pair with the same color, i.e., the underlying graph of H^+ is bipartite. We also observe that in H^+ there is an arc from (u, v) to (u', v') if and only if there is an arc from (v', u') to (v, u). We call this property the *skew - symmetry* of H^+. Note that if $(x, y) \in S$ then $(y, x) \in S'$, the dual component of S. We note that if (u, v) and (v, u) both lie in a strong component of H^+ then H does not have a desired ordering. If we place u before v in an ordering then by following a path from (u, v) to (v, u) in H^+ we conclude that we must place v before u. On the other hand if we place u before v in an ordering then by following a path from (v, u) to (u, v) in H^+ we conclude that v must be placed before u. These imply that if $S = S'$ for some non-trivial component S then H is not an interval bigraph.

In the pair-digraph each strong component S corresponds to a partial order on the vertices of H. This means that for every $(x, y) \in S$ we intend to put x before y in the final ordering. We must include the partial order defined by S or the partial order defined by dual of S denoted by S' (dual contains the reverse pairs). From each pair of strong component S and S' exactly one of them has to be chosen. Therefore we transform the problem of finding the desired ordering to choosing the strong components of a pair-digraph H^+ without creating a circuit (conflict).

A broad overview of the algorithm :

We start with simple definition of implication and transitive closure. For subset R of the vertices of H^+, R^* is the implication closure of R, if every pair (u, v) in R^* is either a pair in R or it is implied by a pair in R (there is in H^+ an arc from a pair in R to (u, v)).

The envelope of R, denoted by $\widehat{R}$, is the smallest set of vertices that contains R and is closed under transitivity and implication. (After applying implication closure some pairs are created and then we apply transitivity to get new pairs and so on).

A sequence $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ of vertices of H^+ is called a *circuit* of H^+. A strong component in H^+ is called non-trivial if it contains more than one pair.

The algorithm consists of the following four main steps.

Step 1. Construct the pair-digraph H^+ and compute its strong components. If there exists a non-trivial component S such that $S = S'$ then report H is not an interval bigraph.

Step 2. We consider the non-trivial components only. Set $D = \emptyset$. From each pair of non-trivial coupled components S and S' we add one of the S^* or $(S')^*$ into D. If by adding S^* into D we do

not encounter a circuit in D then remove S' from further consideration at this step. Otherwise remove S^* from D and add $(S')^*$ into D. If in this case we encounter a circuit then report that H is not an interval bigraph.

If we encounter a circuit at Step (2) then we obtain a set of obstructions known as exobiclique (See Section 3). We prove the following lemma.

Lemma 1.1 *If at step (2) for some non-trivial component S, $D \cup S^*$ contains a circuit and $D \cup (S')^*$ contains a circuit then H contains an exobiclique and hence H is not an interval bigraph.*

Step 3. Set $D = \widehat{D}$. This process can be viewed as a sequence of implications and transitive closures. Consider the first time we encounter a circuit C in $\widehat{D}$. We show that the length of this circuit is at most four. Surprisingly there exists a non-trivial component S (chosen at step (2)) in D associated with circuit C, such that S^* must be removed from D and should be replaced by $(S')^*$ otherwise we still encounter a circuit in the envelope of D. This means that if we keep S in D and reverse some of the other non-trivial components (at step (2)) then after computing the envelope of the new D we still encounter a circuit. Such a component S is called a *dictator* component.

A subset D_1 of the vertices in H^+ is called *complete* if for every pair of non-trivial components S, S' exactly one of them is in D_1. We show the following lemma.

Lemma 1.2 *If we encounter a circuit C during the computation of $\widehat{D}$ then there exists a dictator component S associated to C, such that the envelope of every complete set D_1, containing S has a circuit.*

Step 4. For every dictator component S detected in step (3), delete S^* from D and add $(S')^*$ instead. Keep the rest of non-trivial components from step (2) unchanged. Now compute the envelope of the new set D and if we encounter a circuit at this time, then H is not an interval bigraph.

In order to prove the correctness of the algorithm we need to decompose H based on the strong component in H^+. A deep analysis of the implication closure and transitive closure is needed in order to state the properties of a dictator component. Most of the selections are arbitrary as long as we do not encounter a circuit. An elementary version of pair-digraph was used to obtain an ordering characterization for interval graphs [12]. However what is needed here is a completely different algorithm and it requires different setting and analysis. In fact we take into account the transitive closure and implication closure in our algorithm. We believe pair-digraph can provides a general framework that can be used for solving different problems dealing with ordering characterization of graphs and digraphs with forbidden patterns.

The paper is organized as follows. In Section 2 the basic property of H^+ is presented. H^+ has a special structure; there is no directed path between two different non-trivial component. In Section 3 we consider the structure properties of H and in fact we obtain a decomposition of the H based on the strong component of H^+. If there exist three vertices a, b, c of H such that all the pairs $(a, b), (b, a), (b, c), (c, b), (a, c), (c, a)$ are all in different non trivial components then

we make a module decomposition of H. This decomposition is essential in proving the Lemma 1.1. In section 4 we present the Algorithm. In this section we explain how to obtain dictator component correspond to a circuit in envelope of a complete set D. In fact once a circuit is form for the first time we need to trace back the shortest path of implication and transitive closure that leads to this circuit. In Section 5 we prove the correctness of step 2 of the Algorithm and we prove Lemma 1.1. In Section 6 the structure of a circuit is investigated and we show the length of a circuit is 4 and we decompose each pair of the circuit C in order to find the dictator component for such a circuit. In Section 7 we prove the correctness of the Step 3,4,5. Finally there are some other pairs that must be chosen. These pairs are not in the envelope of the D and we show they can be safely added into envelope of D. In fact this part is similar to the algorithms in paper [12]. The correctness of step (6) is discussed in Section 8. In Section 8 we explain how to construct the pair-digraph in $O(n(m+n))$ time and we discuss the complexity of the algorithm. Finally in Section 9 and 10 we give a construction of some of the obstructions together with examples.

2 Basic properties

We note that a bigraph is an interval bigraph if and only if each connected component is an interval bigraphs. In the remainder of the paper, we shall assume that H is a connected bigraph, with a fixed bipartition (B, W), and that H^+ is the pair-digraph of H.

In [11], it was shown that a bigraph H is an interval bigraph if and only if its vertices admit a linear ordering $<$ without any of the forbidden patterns in Figure 1. If $v_a < v_b < v_c$ and v_a, v_b have the same color and opposite to the color of v_c then $v_a v_c \in E(H)$ implies that $v_b v_c$ is an edge of H.

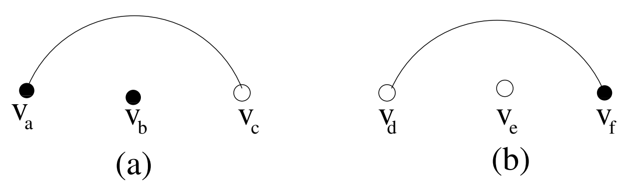

Figure 2: Forbidden Patterns

For two vertices $x, y \in V(H^+)$ we say x dominates y or y is dominated by x and we write $x \to y$, if there exists an arc (directed edge) from x to y in H^+.

Lemma 2.1 *Suppose $<$ is an ordering of H without the forbidden patterns. If $u < v$ and (u, v) dominates (u', v') in H^+, then $u' < v'$.* $\diamond$

Proof: Suppose $(u, v) \to (u', v')$. Now according to the definition of H^+ one of the following happens:

1. $u = u'$ and u, v have different colors and $vv' \in E(H)$ and $uv \notin E(H)$.
2. $v = v'$ and u, v have the same color and $uu' \in E(H)$ and $vu' \notin E(H)$.

If $u < v$ and (1) happens then because uv is not an edge and vv' is an edge we must have $u < v'$. Otherwise we obtain a forbidden pattern in the ordering. If $u < v$ and (2) happens then because $u'u$ is an edge and $u'v$ is not an edge we must have $u' < v$. Otherwise we obtain a forbidden pattern in the ordering. $\diamond$

For a subset S of vertices of H^+, denote

$$S' = \{(u, v) : \ (v, u) \in S\}.$$

For simplicity, we shall also use S to denote the subdigraph H^+ induced by S.

The skew-symmetry of H^+ implies the following fact.

Lemma 2.2 *If S is a strong component of H^+, then so is S'.*

In general, we shall write briefly *component* for *strong component*. Thus either both S and S' are components of H^+ or neither is, and in the former case they are referred to as *coupled components* of H^+. Note that the coupled components S and S' are either equal or disjoint; in the former case we say that S is a *self-coupled* component.

Lemma 2.3 *If H^+ contains a self-coupled component, then H is not an interval bigraph.*

Proof: Let (u, v) be any vertex of S. Since there is a directed walk from (u, v) to (v, u), Lemma 2.1 implies no linear ordering $<$ of $V(H)$ can have $u < v$. Since there is also a directed walk from (v, u) to (u, v), no linear ordering $<$ of $V(H)$ can have $u > v$ either, and hence there is no linear ordering. $\diamond$

A sequence $(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, x_n), (x_n, x_0)$ of vertices of H^+ is called a *circuit* of H^+. If a strong component S of H^+ contains both $(u, v), (v, u)$ then there is a circuit with $n = 1$. It is clear that if a component of H^+ contains any circuit, then H is not an interval bigraph.

A similar line of reasoning shows the following fact.

Lemma 2.4 *Suppose that H^+ contains no self-coupled components, and let D be any subset of $V(H^+)$ containing exactly one of each pair of coupled components. Then D is the set of arcs of a tournament on $V(H)$.*

Moreover, such a D can be chosen to be a transitive tournament if and only if H is an interval bigraph.

We shall say two edges ab, cd of H are *independent* if the subgraph of H induced by the vertices a, b, c, d has just the two edges ab, cd. Note that if ab, cd are independent edges in H then the component of H^+ containing the pair (a, c) also contains the pairs $(a, d), (b, c), (b, d)$. Moreover, if a and c have the same color in H, the pairs $(a, c), (b, c), (b, d)$,
(a, d) form a directed four-cycle in H^+ in the given order; and if a and c have the opposite color, the same vertices form a directed four-cycle in the reversed order. In any event, an independent pair of edges yields at least four vertices in the corresponding component of H^+. Conversely we have the following lemma.

Lemma 2.5 *Suppose S is a non-trivial component of H^+ containing the vertex (u,v). Then there exist two independent edges uu', vv' of H, and hence S contains at least the four vertices $(u,v),(u,v'),(u',v),(u',v')$.*

Proof: Since S is non-trivial, (u,v) dominates some vertex of S and is dominated by some vertex of S. First suppose u and v have the same color in H. Then (u,v) dominates some $(u',v) \in S$ and is dominated by some $(u,v') \in S$. Now uu', vv' must be edges of H and $uv, uv', u'v, u'v'$ must be non-edges of H. Thus uu', vv' are independent edges in H. Now suppose u and v have different colors. We note that (u,v) dominates some $(u,v') \in SS$ and hence uv is not an edge of H and vv' is an edge of H. Since (u,v') dominates some pair $(u',v') \in S$, uu' is an edge and $u'v'$ is not an edge of H. Now uu', vv' are edges of H and $uv, uv', u'v, u'v'$ must be non-edges of H. Thus uu', vv' are independent edges in H. ◇

Thus a non-trivial component of H^+ must have at least four vertices; for convenience, we shall call such components *non-trivial components*. Recall that any pair (u,v) in a non-trivial component of H^+ must have u and v non-adjacent in H.

Let R be a subset of vertices of H^+. The *implication closure* of R, denoted R^*, consists of all vertices (u,v) of H^+ such that either $(u,v) \in R$ or (u,v) is dominated by some $(u',v') \in R$. We say (u,v) is *implied* by R if $(u,v) \in R^* - R$.

The structure of components of H^+ is quite special; the giant and trivial components interact in simple ways. A trivial component will be called a *source component* if its unique vertex has indegree zero, and a *sink component* if its unique vertex has out-degree zero. Before we describe the structure, we establish a useful counterpart to Lemma 2.5.

Lemma 2.6 *A pair (a,c) is implied by a non-trivial component of H^+ if and only if H contains an induced path a,b,c,d,e, such that $N(a) \subset N(c)$. If such a path exists, then the non-trivial component S implying (a,c) contains all the pairs $(a,d),(a,e),(b,d),(b,e)$.*

Proof: If such a path exists, then ab, de are independent edges and so the pairs $(a,d),(a,e)$, $(b,d),(b,e)$ lie in a non-trivial component by the remarks preceding Lemma 2.5. Moreover, H^+ contains the arc from (a,d) to (a,c), so that (a,c) is indeed implied by this non-trivial component.

To prove the converse, suppose (a,c) is implied by a non-trivial component S. We first observe that the colors of a and c must be the same. Otherwise, say, a is black and c is white, and there exists a white vertex u such that the pair (u,c) is in S and dominates (a,c). By Lemma 2.5, there would exist two independent edges uz, cy. Looking at the edges and non-edges amongst u,c and a,z,y, we see that H^+ contains the arcs

$$(u,c) \to (a,c) \to (a,y) \to (u,y).$$

Since both (u,c) and (u,y) are in S, the pair (a,c) must also be in S, contrary to what we assumed.

Therefore a and c must have the same color in H, say black. In this case there exists a white vertex d such that $(a,d) \in S$ dominates (a,c) in H^+. Hence d is adjacent in H to c but not to a. If there was also a vertex t adjacent in H to a but not to c, then at, cd would be independent edges, placing (a,c) in S. Thus every neighbor of a in H is also a neighbor of c in H. Finally, since (a,d) is in a non-trivial component S, Lemma 2.5 yields vertices b,e such that ab, de are independent edges in H. It follows that a,b,c,d,e is an induced path in H. ◇

We emphasize that ab, de from the last Lemma are independent edges. The inclusion $N(a) \subset N(c)$ implies the following Corollary.

Corollary 2.7 *If there is an arc from a non-trivial component S of H^+ to a vertex $v \notin S$ then v forms a trivial component of S which is a sink component; if there is an arc to a non-trivial component S of H^+ from a vertex $v \notin S$, then v forms a trivial component of S which is a source component.* $\diamond$

In particular, we note that H^+ has no directed path joining two non-trivial components. To give even more structure to the components of H^+, we recall the following definition. The *condensation* of a digraph D is a digraph obtained from D by identifying the vertices in each component and deleting loops and multiple edges.

Lemma 2.8 *Every directed path in the condensation of H^+ has at most three vertices.*

Proof: If a directed path P in the condensation of H^+ goes through a vertex corresponding to a non-trivial component S in H^+, then P has at most three vertices by Corollary 2.7. If P contains only vertices in trivial components, suppose (x, y) is a vertex on P which has both a predecessor and a successor on P. If x and y have the same color in H, then the successor is some (x', y) and the predecessor is some (x, y'); this would mean that xx', yy' are independent edges contradicting the fact that P is a path in the condensation of H^+. Thus x and y have the opposite color in H, and the successor of (x, y) in P is some (x, y') and the predecessor is some (x', y). Thus (x, y) is not an edge of H, whence (x', y') must be an edge of H, otherwise we would have independent edges xx', yy' and conclude as above. By the same reasoning, every vertex adjacent to x is also adjacent to y', and every vertex adjacent to y is also adjacent to x'. This implies that (x', y) has indegree zero, and (x, y') has outdegree zero, and P has only three vertices. $\diamond$

3 Structures

An *exobiclique* in a bigraph H with bipartition (B, W) is a biclique with nonempty parts $M \subseteq B$ and $N \subseteq W$ such that both $B \setminus M$ and $W \setminus N$ contain three vertices with incomparable neighborhoods.

The following result is proved in [11].

Theorem 3.1 *If H contains an exobiclique, then H is not an interval bigraph.* $\diamond$

We say that a bigraph H with bipartition (B, W) is a *pre-insect*, if the vertices of H can be partitioned into subgraphs $H_1, H_2, \ldots, H_k, X, Y, Z$, where $k \geq 3$ and the following properties are satisfied:

(1) each H_i is a non-trivial component of $H' = H \setminus X \setminus Y \setminus Z$;

(2) X is a complete bipartite graph;

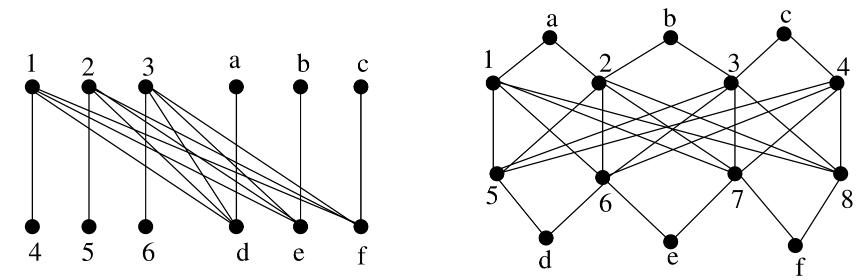

Figure 3: Exobicliques

(3) every vertex of X is adjacent to all vertices of opposite color in H';

(4) there are no edges between Y and H';

(5) there is no edge ab in Y such that both a and b are adjacent to all vertices in X of opposite color;

(6) if Z is non-empty, then either

(i) every vertex of Z is adjacent to all vertices of opposite color in each H_i with $i > 1$, or

(ii) every vertex of Z is adjacent to at least one vertex of opposite color in each H_i with $i > 1$, and there are no edges between Z and H_1;

(7) every vertex of Z is adjacent to all vertices of opposite color in $X \cup Z$.

We make the following observation on the non-trivial components of a pre-insect.

Remark 3.1 *If H is a pre-insect, all pairs (u, v) where $u \in H_i$ and $v \in H_j$, for some fixed $i \neq j$, are contained in the same non-trivial component $S^{(i,j)}$ of H^+. If Z is not empty, we moreover have $S^{(1,2)} = S^{(1,3)} = \ldots = S^{(1,k)}$. Otherwise $(i, j) \neq (i', j')$ implies that $S^{(i,j)}$ and $S^{(i',j')}$ are distinct components of H^+.*

In the sequel, we shall use S_{uv} to denote the component of H^+ containing the vertex (u, v). Thus S_{uv} and S_{vu} are coupled components of H^+.
We shall say that a vertex v is *completely adjacent* to a subgraph V of H if v is adjacent to every vertex of opposite color in V. We shall also say that v is *completely non-adjacent* to V if it has no edges to V.

Theorem 3.2 *Suppose that H^+ has no self-coupled components.*

If H has three vertices u, v, w such that S_{uv}, S_{vw} are giant components of H^+ and $S_{uv} \neq S_{vw}, S_{uv} \neq S_{wv}$, then H is a pre-insect and u, v, w belong to different connected components of H'.

Moreover, in this case $S_{wu} \neq S_{uv}, S_{wu} \neq S_{vw}$. If all $S_{uv}, S_{vu}, S_{vw}, S_{wv}, S_{uw}, S_{wu}$ are pairwise distinct then the subgraph Z is empty; otherwise Z is non-empty and either $S_{uw} = S_{uv}$ or $S_{uw} = S_{vw}$.

Proof: First we observe that the skew-symmetry of H^+ implies that $S_{uv} \neq S_{vw}, S_{uv} \neq S_{wv}$ also yields $S_{vu} \neq S_{wv}, S_{vu} \neq S_{vw}$. So we may freely use any of these properties in the proof.

Since S_{uv} is a non-trivial component, by Lemma 2.5, there are two independent edges uu', vv'; similarly, there are two independent edges vv'', ww'. Assume that u, v, w are of the same color - in case when u, v are of different colors, we switch the names of u, u' and when v, w are of different colors, we switch the names of w, w'.

Since H^+ has no self-coupled components, we have that $S_{uv}, S_{vu}, S_{vw}, S_{wv}$ are pairwise distinct giant components of H^+. Hence by Corollary 2.7 there is no directed path in H^+ between any two of them.

We claim that uu', ww' are independent edges. Indeed, an adjacency between u and w' in H would mean an arc from (u, v) to (w', v) in H^+ and an adjacency between u' and w in H would mean a directed edge from (w, v) to (u', v), both contradicting our assumptions. It follows that S_{uw} and S_{wu} are also non-trivial components.

If both uv'' and wv' are edges of H, then there is an arc from (u, v') to (u, w), implying $S_{uv} = S_{uw}$ and there is an arc from (v'', w') to (u, w') implying that $S_{vw} = S_{uw}$, and hence $S_{uv} = S_{vw}$, a contradiction. So either uv'' or wv' is not an edge of H. By symmetry, we may assume that wv' is not an edge of H. Hence uu', vv', and ww' are three pairwise independent edges of H.

Let S be a maximal induced subgraph of H which consists of three connected components H_1, H_2, H_3 containing uu', vv', ww' respectively. Let X be the set of vertices completely adjacent to $H_1 \cup H_2 \cup H_3$. Let Y' be the set of vertices completely non-adjacent to $H_1 \cup H_2 \cup H_3$, and let T be the subset of Y' consisting of vertices completely adjacent to X. We shall also use X, Y', T, etc., to denote the subgraphs of H induced by these vertex sets.

We let H' consist of H_1, H_2, H_3 and all non-trivial connected components $H_4, \ldots, H_k$ of T. We also let $Y = Y' \setminus H'$, and let $Z = H \setminus (H' \cup X \cup Y)$. We now verify the conditions (1-7).

It follows from the definition that every vertex of X is completely adjacent to H', every vertex of Y is completely non-adjacent to H', and every vertex of Z has neighbors from at least two of H_1, H_2, H_3 (but is not completely adjacent to $H_1 \cup H_2 \cup H_3$).

We claim that X is a complete bigraph. Indeed, suppose that x, x' are vertices of X of opposite colors, where x is of the same color as u. If x, x' are not adjacent, then $(u', v), (u', x'), (x, x'), (x, v), (w', v)$ is a directed path in H^+ from S_{uv} to S_{wv}, a contradiction.

The definition of H' also implies that if yy' is an edge of Y, then y, y' cannot both be completely adjacent to X.

Let $a \in H_1, b \in H_2, c \in H_3$ be three vertices of the same color. Suppose that some z is adjacent to two of these vertices but not to the third one; say, z is adjacent to b and c but not to a. Clearly, $z \in Z$. Let a' be any vertex in H_1 adjacent to a. Then $(a', b), (a', z), (a, z), (a, c)$ is a directed path from S_{uv} to S_{uw}, implying $S_{uv} = S_{uw}$. This property implies that if $S_{uv}, S_{vu}, S_{vw}, S_{wv}, S_{uw}$, and S_{wu} are pairwise distinct then Z is empty. (The converse is also true, i.e., if Z is empty then $S_{uv}, S_{vu}, S_{vw}, S_{wv}, S_{uw}$, and S_{wu} are pairwise distinct.)

Since $S_{uv} \neq S_{vw}, S_{wv}$, the same property implies that every vertex of Z adjacent to vertices in H_1 and in H_3 must be completely adjacent either to $H_1 \cup H_2$ or to $H_2 \cup H_3$. If some vertex

of $z \in Z$ is completely adjacent to $H_2 \cup H_3$, then z is not completely adjacent to H_1 and hence the above property implies $S_{uv} = S_{uw}$; similarly, if some vertex of Z is completely adjacent to $H_1 \cup H_2$, then we have $S_{wu} = S_{wv}$ (i.e., $S_{uw} = S_{vw}$). Since $S_{uv} \neq S_{vw}$, Z cannot contain both a vertex completely adjacent to $H_1 \cup H_2$ and a vertex completely adjacent to $H_2 \cup H_3$. Therefore, when Z is not empty, either H_1 or H_3 enjoys a "special position", in the sense that

- each vertex of Z is adjacent to at least one vertex in H_2 and at least one vertex in H_3 and is nonadjacent to at least one vertex in H_1. Moreover, if it is also adjacent to a vertex in H_1, then it is completely completely adjacent to $H_2 \cup H_3$. (This corresponds to the case $S_{uw} = S_{uv}$.)
- each vertex of Z is adjacent to at least one vertex in H_1 and at least one vertex in H_2 and is nonadjacent to at least one vertex in H_3. Moreover, if it is also adjacent to a vertex in H_3, then it is completely completely adjacent to $H_1 \cup H_2$. (This corresponds to the case $S_{uw} = S_{vw}$.)

In either case, we have $S_{wu} \neq S_{uv}, S_{wu} \neq S_{vw}$.

Finally, we show that every vertex of Z is completely adjacent to $X \cup Z$. Let $z \in Z$. From above we know that either z has neighbors in H_1 and in H_2, or z has neighbors in H_2 and in H_3. Assume that $a' \in H_1$ and $b' \in H_2$ are neighbors of z. (A similar argument applies in the other case.) Suppose that z is not adjacent to a vertex $x' \in X$ of the opposite color. Let $a \in H_1$ and $b \in H_2$ be adjacent to a' and b' respectively. Since each vertex of X is completely adjacent to $H_1 \cup H_2$, the vertex x' is adjacent to both a and b. Thus $za'ax'bb'z$ is an induced 6-cycle in H, which is easily seen to imply that $S_{uv} = S_{vu}$, a contradiction. Suppose now that z is not adjacent to a vertex $z' \in Z$ of opposite color. Then as above z' has neighbors $a \in H_1$ and $b \in H_2$. Choose such vertices a, a', b, b' so that a, a' have the minimum distance in H_1 and b, b' have the minimum distance in H_2. It is easy to see that there is an induced cycle of length at least six in H, using vertices z, a, a', b, b' a shortest path in H_1 joining a, a' and a shortest path in H_2 joining b, b'. This implies again that $S_{uv} = S_{vu}$, a contradiction. $\diamond$

We now consider the possibility that for some three vertices u, v, w of H, the components S_{uv}, S_{vw} coincide; of course then this common component $S_{uv} = S_{vw}$ is a non-trivial component.

Lemma 3.3 *Suppose that H^+ has no self-coupled components. If for some three vertices u, v, w of H we have $S_{uv} = S_{vw}$, then we also have $S_{uv} = S_{uw}$.*

Proof: Since S_{uv} is a non-trivial component, there are independent edges uu', vv'; similarly, there are independent edges vv'', ww'. We may assume that u, v, w are of the same color - in case when u, v are of different colors, we switch the names of u, u' and similarly for v, w.

We claim that neither uw' nor wu' is an edge of H. Indeed, if uw' is an edge of H, then uw', vv' are independent edges of H, which implies that $S_{uv} = S_{w'v} = S_{wv}$. However, we know by assumption $S_{uv} = S_{vw}$. Thus $S_{wv} = S_{vw}$, a contradiction. Similarly, if wu' is an edge, then wu', vv'' are independent edges, which implies $S_{vw} = S_{vu'} = S_{vu}$. Since $S_{uv} = S_{vw}$, we have $S_{uv} = S_{vu}$, a contradiction.

If uv'' and wv' are both edges of H, then they are independent and we have $S_{uv} = S_{uv'} = S_{uw}$. By symmetry, we may assume that wv' is not an edge of H. Hence we obtain three pairwise independent edges uu', vv', ww' of H.

Following the proof of Theorem 3.2, we define the subgraphs H', X, Y, Z. Since $S_{uv} = S_{vw}$, the set Z is not empty. Each vertex of Z has neighbors in at least two of H_1, H_2, H_3 but is not completely adjacent to $H_1 \cup H_2 \cup H_3$. It is not possible that some vertex of Z is adjacent to vertices in H_1 and in H_3 but nonadjacent to a vertex in H_2, as otherwise we would have $S_{uv} = S_{wv}$. Since $S_{uv} = S_{vw}$, $S_{wv} = S_{vw}$, a contradiction. If some vertex of Z adjacent to vertices in H_2 and in H_3 but nonadjacent to a vertex in H_1, then $S_{uv} = S_{uw}$; similarly, if some vertex adjacent to vertices in H_1 and in H_2 but nonadjacent to a vertex in H_3, then $S_{uw} = S_{vw}$. This completes the proof. $\diamond$

We now summarize the possible structure of the six related components $S_{uv}, S_{vu}, S_{vw}, S_{wv}, S_{uw}$, and S_{wu}. Theorem 3.2 and Lemma 3.3 imply the following corollary.

Corollary 3.4 *Suppose that H^+ has no self-coupled components.*

Let u, v, w be three vertices of H such that S_{uv} and S_{vw} are non-trivial components of H^+. Then S_{uw} is also a non-trivial component of H^+.

Moreover, one of the following occurs, up to a permutation of u, v, w.

(i) $S_{uv}, S_{vu}, S_{vw}, S_{wv}, S_{uw}$, *and* S_{wu} *are pairwise distinct;*

(ii) $S_{uv} = S_{uw}$, $S_{wu} = S_{vu}$, S_{vw}, S_{wv} *are pairwise distinct;*

(iii) $S_{uv} = S_{vw} = S_{uw}$ *and* $S_{vu} = S_{wv} = S_{wu}$ *are distinct.*

$\diamond$

4 The Recognition Algorithm

We now present our algorithm for the recognition of interval bigraphs. During the algorithm, we maintain a subdigraph D of H^+, Initially, D is empty ; at successful termination, D will be a transitive tournament as described in Lemma 2.4.

Let R be a set of vertices of H^+. The *envelope* of R, denoted $\widehat{R}$, is the smallest set of vertices that contains R and is closed under transitivity and implication. For the purposes of the proofs we visualize taking the envelope of R as divided into consecutive *levels*, where in zero-th level we just replace R by its implication closure, and in each subsequent level we replace R by the implication closure of its transitive closure. The pairs in the envelope of R can be thought of as forming a digraph on $V(H)$, and each pair can be thought of as having a label corresponding to its level. The arcs in R, and those implied by R have the label 0, arcs obtained by transitivity from the arcs labeled 0, as well as all arcs implied by them have label 1, and so on.

Note that $R \subseteq R^* \subseteq \widehat{R}$ and each of $R, R^*, \widehat{R}$ may or may not contain a circuit.

Lemma 4.1 *Let S be a non-trivial component, and S' its coupled component. If both $\widehat{S}$ and $\widehat{S'}$ contain a circuit, then H is not an interval bigraph.*

Proof: By Lemma 2.1, if $\widehat{S}$ contains a circuit, S should not form a part of D. Now Lemma 2.4 yields a contradiction.

Definition 4.2 *Let $\mathcal{R} = \{R_1, R_2, ..., R_k, S\}$ be an ordered set of non-trivial components in D that $\widehat{\mathcal{R}}$ contains a circuit $C = (x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$. We say S is a* dictator *for C if the envelope of ordered set $\mathcal{R}' = \{R'_1, R'_2, ..., R'_t, R_{t+1}, ..., R_k, S\}$, for some $t \leq k$, also contains a circuit C'. In fact by reversing some of the R_i's, and taking the envelope of the new set we still get a circuit.*

Definition 4.3 *A set D_1 is called complete if for every pair of non-trivial components S, S' exactly one of them is in D_1.*

Equivalent definition of a dictator component S is as follows.

Definition 4.4 *We say a component S is a dictator component if envelope of every complete set D with $S \subset D$, contains a circuit.*

For the purpose of the algorithm once a pair (x, y) is created we associate the time (level) to (x, y). Let $T(x, y)$ be the level in which (x, y) is created.

Each pair (x, y) carries a dictator code that essentially shows the dictator component involved in creating a circuit containing (x, y).

(a) If $(x, y) \in S^*$ for some non-trivial component S then $Dict(x, y) = S$.

(b) If x, y have different colors and (x, y) is implied by some pair (u, y) then $Dict(x, y) = Dict(u, y)$.

(c) If x, y have the same color and (x, y) is implied by some pair (x, w) then $Dict(x, y) = Dict(x, w)$.

(d) If x, y have the same color and (x, y) is by transitivity on $(x, w), (w, y)$ then $Dict(x, y) = Dict(w, y)$.

(e) If x, y have different colors and (x, y) is by transitivity on $(x, w), (w, y)$ then $Dict(x, y) = Dict(x, w)$.

Consider a complete set D after Step (2). A pair (x, y) is called *original* if at least one of the $(x, y), (y, x)$ is not in D and if (x, y) is implied by another pair $(x', y') \in D$ then (x', y') is original and if (x, y) is by transitivity with the pairs $(x, w), (w, y) \in D$ then both (x, w) and (w, y) are original. During the computation of $\widehat{D}$ we consider the circuits created by the original pairs.

Algorithm

Input: A connected bigraph H with a bipartition (B, W).

Output: An interval representation of H or a claim that H is not an interval bigraph.

1. Construct the pair-digraph H^+ of H, and compute its components; if any are self-coupled report that H is not an interval bigraph.

2. For each pair of coupled non-trivial components S, S', add one of S^* and $(S')^*$ to D as long as it does not create a circuit, and delete the other one from further consideration in this step. If neither S^* nor S'^* can be added to D without creating a circuit, then report that H is not an interval bigraph.

3. Add the created pairs during the computation of $\widehat{D}$ one by one into D. If by adding an *original* pair (x, y) into D we close a circuit then add $Dict(x, y)$ into set $\mathcal{DT}$.

4. Let $D_1 = \emptyset$. For every $S \in \mathcal{DT}$ add $(S')^*$ into D_1. For every non-trivial component $R \in D \setminus \mathcal{DT}$ add R^* into D_1.

5. Set $D_1 = \widehat{D_1}$. If there is a circuit in D_1 then report H is not an interval bigraph.

6. As long as there remain (trivial) components not in D_1, add the unique vertex of a sink component (sink in the remaining subdigraph of H^+) to D_1 and remove its coupled component from further consideration.

7. Let $u < v$ if $(u, v) \in D_1$, yielding an ordering of $V(H)$ without the forbidden patterns from Figure 1; obtain the corresponding interval representation of H as described in [11].

End Algorithm

The purpose of introducing the original pair is to detect all the dictator components in one run of computing $\widehat{D}$; the envelope of D.

In section 6 we show that if a circuit happens then its length is exactly 4 and we can identify a dictator component associated to this circuit by using $Dict(x, y)$ where (x, y) is a pair of the circuit. We also need to observe that some pair (x, y) might be implied by different pairs (x, w) and (x, w') such that they are both created at the same level but we show in Section (3), (The structure of the circuit) that this is not relevant and if (x, y) is involved in a first created circuit then $Dict(x, w) = Dict(x, w')$ are the same.

Let $C = (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ be a circuit such that x_0, x_3 have the same color opposite to the color of x_1, x_2.
Each pair (x_i, x_{i+1}), $0 \leq i \leq 3$ is an implied pair or inside a non-trivial component. No pair (x_i, x_{i+1}) is by transitivity, (the sum is taken module 3).

Definition 4.5 *A pair $(x, y) \in D$ is* complex *if it is not in S^* for any non-trivial component S. Otherwise (x, y) is a* simple *pair.*

For more clarification we show the following in Section 6.

1. If (x_i, x_{i+1}) is a complex pair and (x_{i+1}, x_{i+2}) for $i = 0, 2$ is an implied pair by non-trivial component S then S is a dictator component.

2. For $i = 1, 3$, if (x_i, x_{i+1}) is a simple pair in a non-trivial component S and (x_{i-1}, x_i) is a complex pair then S is a dictator component.

3. If both $(x_i, x_{i+1}), (x_{i+1}, x_{i+2})$ are complex pairs then the dictator component for this circuit would be $Dict(x_i, x_{i+1})$.

To prove the correctness of the algorithm we first observe that Step 1 is justified by Lemma 2.3. In the next section we focus on proving the correctness of Step 2.

5 Correctness of Step 2

We consider what happens when a circuit is formed during the execution of Step 2 of our algorithm; our goal is to prove that in such a case H contains an exobiclique and hence is not an interval bigraph. Note that we only get to Step 2 if H^+ has no self-coupled components, so we do not need to explicitly make this assumption.

Theorem 5.1 *Suppose that within Step 2 we have so far constructed a D without circuits, and then for the next non-trivial component S we find that $D \cup S^*$ has circuits. Let C : $(x_0, x_1), (x_1, x_2), \ldots, (x_n, x_0)$ be a shortest circuit in $D \cup S^*$. Then one of the following must occur.*

- *(i) H is a pre-insect with empty Z, each x_i belongs to some subgraph H_{a_i}, and $i \neq j$ implies $a_i \neq a_j$, or*
- *(ii) H is a pre-insect with non-empty Z, each x_i belongs to some subgraph H_{a_i} with $i > 1$, and $i \neq j$ implies $a_i \neq a_j$, or*
- *(iii) H contains an exobiclique.*

Proof: From the way the algorithm constructs D, we know that each pair (x_i, x_{i+1}) either belongs to or is implied by a non-trivial component in $D \cup S^*$. The length of C is at least three, i.e., $n \geq 2$, otherwise $S_{x_0x_1}$ and $S_{x_1x_0}$ are both in $D \cup S^*$, contrary to our algorithm.

We first show that no two consecutive pairs of C are both implied by non-trivial components. Indeed, suppose that for some subscript s, both (x_{s-2}, x_{s-1}) and (x_{s-1}, x_s) are implied by non-trivial components. Then by Lemma 2.6, there are induced paths $x_{s-2}, x, x_{s-1}, y, z$ and x_{s-1}, u, x_s, v, w with $N(x_{s-2}) \subset N(x_{s-1}) \subset N(x_s)$. Since x, y are adjacent to x_{s-1}, they are adjacent also to x_s. Thus x_{s-2}, x, x_s, y, z is an induced path in H (with $N(x_{s-2}) \subset N(x_s)$). By Lemma 2.6, (x_{s-2}, x_s) is implied by $S_{x_{s-2}y}$. We know that $S_{x_{s-2}y}$ is in $D \cup S^*$ because it implies (x_{s-2}, x_{s-1}). Hence (x_{s-2}, x_s) is also in $D \cup S^*$. Replacing $(x_{s-2}, x_{s-1}), (x_{s-1}, x_s)$ with (x_{s-2}, x_s) in C, we obtain a circuit in $D \cup S^*$ shorter than C, a contradiction.

Suppose that for some s both (x_{s-2}, x_{s-1}) and (x_{s-1}, x_s) belong to non-trivial components. By Lemma 3.3, (x_s, x_{s-2}) also belongs to a non-trivial component. Consider $S_{x_{s-2}x_{s-1}}, S_{x_{s-1}x_s}$, and $S_{x_sx_{s-2}}$. Suppose that any two of these are equal. Then they are equal to the component coupled with the third one, by Lemma 3.3. This means that either (x_{s-1}, x_{s-2}), or (x_s, x_{s-1}), or (x_{s-2}, x_s) is contained in $D \cup S^*$, each resulting in a shorter circuit, and a contradiction. Therefore, by Corollary 3.4, we have the following cases:

(1) the six components $S_{x_{s-2}x_{s-1}}, S_{x_{s-1}x_{s-2}}, S_{x_{s-1}x_s}, S_{x_sx_{s-1}}, S_{x_sx_{s-2}}, S_{x_{s-2}x_s}$ are pairwise distinct;

(2) $S_{x_{s-2}x_{s-1}} = S_{x_{s-2}x_s}$;

(3) $S_{x_{s-2}x_{s-1}} = S_{x_s x_{s-1}}$; or

(4) $S_{x_{s-1}x_s} = S_{x_{s-2}x_s}$.

Since (2), (3), and (4) result in a circuit in $D \cup S^*$ shorter than C, we must have (1). By Theorem 3.2, H is a pre-insect with empty set Z. So we either have each x_i is in H', implying the case (i), or some x_j belongs to $X \cup Y$. As in the proof of Theorem 3.2, let $H_1, H_2, H_3, \dots$ be the connected components of H' where $x_{s-2} \in H_1, x_{s-1} \in H_2, x_s \in H_3$. Without loss of generality assume that $x_{s+1}, \dots, x_{t-1} \in X \cup Y$ and $x_t \in H_d$. Note that $d \neq 3$, by the minimality of C.

We show that $S_{x_{t-1}x_t}$ is a trivial component. Otherwise, by Lemma 2.5, we obtain two independent edges $x_{t-1}u$ and $x_t v$. It is easy to see that $x_{t-1}u$ lies in Y and the vertex v is either in H_d or in X. We assume that x_t is of the same color as x_{t-1} (the discussion is similar when they are of different colors). We know from above that either x_{t-1} or u is not adjacent to some vertex in X of opposite color. Assume first that x_{t-1} is not adjacent to $w \in X$ of opposite color. Since each vertex of X is completely adjacent to H', w is adjacent to x_t and a vertex $w' \in H_3$ (note that H_3 contains x_s). We see now that $x_{t-1}u$ is independent with both wx_t and ww', which means that $S_{x_{t-1}x_t} = S_{x_{t-1}x_s}$. We have a shorter circuit $(x_s, x_{s+1}), \dots, (x_{t-1}, x_s)$, a contradiction. The proof is similar if u is not adjacent to some vertex in X. So $S_{x_{t-1}x_t}$ is a trivial component, and hence (x_{t-1}, x_t) is implied by some non-trivial component.

By Lemma 2.6 there is an induced path $x_{t-1}yx_tzw$ in H such that $N(x_{t-1}) \subset N(x_t)$, which implies that $y \in X$ and $x_{t-1} \in Y$. Clearly, $w \notin X \cup H_d$ as it is not adjacent to $y \in X$ and $z \notin Y$ as it is adjacent to x_t. It follows that $z \in X$ and w is in Y. Note that (x_{t-1}, z) are in a non-trivial component. Now $(x_{t-1}, z) \to (x_{t-1}, v)$ for some $v \in H_2$. If v, x_{s-1} have the same color and in this case $(x_{t-1}, z) \to (x_{t-1}, x_{s-1})$ and hence we a get a shorter circuit. If v, x_{s-1} have different colors then there is also circuit

$$(x_0, x_1), ..., (x_{s-2}, v), (v, x_s), (x_s, x_{s+1}), ..., (x_n, x_0)$$

in D since $(x_{s-2}, v), (x_{s-2}, x_{s-1})$ are in the same non-trivial component, and $(v, x_s), (x_{s-1}, x_s)$ are in a same non-trivial component. Therefore we get a shorter circuit.

In remains to consider the situation where consecutive pairs of C always alternate, in belonging to, and being implied by, a non-trivial component. Suppose that (x_i, x_{i+1}) is implied by a non-trivial component. By Lemma 2.6, there is an induced path $x_i a x_{i+1} b c$ with $N(x_i) \subset N(x_{i+1})$. Note that x_i and x_{i+1} have the same colour.

We show that x_{i+2} has colour different from that of x_i. For a contradiction, suppose that they are of the same colour. Let $x_{i+1}f, x_{i+2}g$ be independent edges in H; such edges exist because (x_{i+1}, x_{i+2}) belongs to a giant component. Since $N(x_i) \subset N(x_{i+1})$ and $x_{i+1}g$ is not an edge of H, also $x_i g$ is not an edge of H. We also see that bx_{i+2} is not an edge, otherwise (x_i, x_{i+2}) would be implied by the non-trivial component $S_{x_i b}$. Since $S_{x_i b}$ is in $D \cup S^*$, the pair (x_i, x_{i+2}) is in $D \cup S^*$, and we obtain a circuit shorter than C. If ax_{i+2} is an edge, then we have $S_{ab} = S_{x_{i+2}b} = S_{x_{i+2}x_{i+1}}$, implying (x_{i+2}, x_i) is in $D \cup S^*$, a contradiction. So ax_{i+2} is not an edge. Hence we have $S_{bg} = S_{x_{i+1}x_{i+2}} = S_{ax_{i+2}} = S_{x_i x_{i+2}}$, a contradiction. Therefore x_i and x_{i+2} have different colours.

Without loss of generality, we may assume that x_i, x_{i+1} have the same colour for each even i. Thus (x_i, x_{i+1}) is implied by a non-trivial component if and only if i is even. We now proceed

to identify an exobiclique in H. Since the arguments are similar, but there are many details, we organize the proof into small steps. Note that by our assumption v_1, v_2 have different colors.

1. Since (v_1, v_2) is in a non-trivial component in H^+, by Lemma 2.5 there are two independent edges v_1a and v_2b.
2. Since (v_2, v_3) forms a trivial component in H^+, by Lemma 2.6 there is an induced path v_2, c, v_3, e, d in H satisfying the property that $N(v_2) \subseteq N(v_3)$. Thus b is adjacent to v_3, as it is adjacent to v_2.
3. We have v_1v_3 an edge of H as otherwise
$$(v_1, v_2) \to (v_1, b) \to (a, b) \to (a, v_3) \to (v_1, v_3)$$
and we have a shorter circuit using (v_1, v_3) instead of $(v_1, v_2), (v_2, v_3)$, a contradiction.
4. Since (v_3, v_4) is in a non-trivial component in H^+, there are independent edges v_3f, v_4g.
5. Since (v_4, v_5) forms a trivial component in H^+, there is an induced path v_4, h, v_5, j, k in H with $N(v_4) \subseteq N(v_5)$. Thus gv_5 is an edge of H, as g is a neighbor of v_4.
6. Now we see av_4 is NOT an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, a) \to (b, a) \to (b, v_1) \to (v_2, v_1),$$
a contradiction.
7. v_1g is an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, g) \to (v_1, g) \to (v_1, v_4),$$
and we have a shorter circuit.
8. v_1h is an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, h) \to (v_1, h) \to (v_1, v_4),$$
again yielding a shorter circuit.
9. v_1j is an edge of H. If $n = 4$ then follows from the fact that v_5j is an edge. Otherwise (v_1, j) and (v_4, j) are in the same non-trivial component of H^+ which implies the pair (v_4, v_5). Since $(v_1, j) \to (v_1, v_5)$, we have a shorter circuit.
10. v_2v_4 is not an edge as otherwise $(v_1, v_2) \to (v_1, v_4)$ and we have a shorter circuit.
11. v_2v_5 is NOT an edge of H, as otherwise $(v_1, v_2) \to (v_1, v_5)$ and we have a shorter circuit.
12. ch is an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, h) \to (c, v_4) \to (v_2, v_4),$$
and we have a shorter circuit.

13. Similar to above item we have cg is an edge of H.

14. cj is an edge of H, as otherwise (h, j) and (v_4, j) are in the same non-trivial component and we have
$$(h, j) \to (c, v_1) \to (v_2, v_1),$$
and we have a shorter circuit.

15. bh is an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, h) \to (b, h) \to (b, v_4) \to (v_2, v_4),$$
and we have a shorter circuit.

16. Similar to above item we have bg is an edge of H.

17. bj is an edge of H, as otherwise
$$(v_1, b) \to (j, b) \to (j, h),$$
while (h, j) is in D, a contradiction to our algorithm.

18. eh is an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, h) \to (e, h) \to (e, c).$$
This is a contradiction as (c, e) is in the non-trivial component which implies (v_2, v_3).

19. ej is an edge of H, as otherwise (e, j) and (v_4, j) are in the same non-trivial component which implies (v_4, v_5). But $(e, j) \to (e, c)$, a contradiction as (e, c) is in the non-trivial component implying (v_2, v_3).

20. eg is an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, g) \to (e, g) \to (e, v_4) \to (j, v_4),$$
a contradiction as $S_{v_4 j}$ is the non-trivial component that implies (v_4, v_5).

21. $v_1 d$ is NOT an edge of H, as otherwise $(v_1, v_2) \to (v_1, c) \to (d, c)$, a contradiction since S_{cd} is the non-trivial component implying (v_2, v_3).

22. ae is NOT an edge of H, as otherwise $(v_1, v_2) \to (e, v_2)$, a contradiction as $S_{v_2 e}$ is the non-trivial component which implies (v_2, v_3).

23. fj is NOT an edge of H, as otherwise
$$(v_3, v_4) \to (v_3, g) \to (f, g) \to (f, v_4) \to (j, v_4),$$
a contradiction as $S_{v_4 j}$ is the non-trivial component implying (v_4, v_5).

24. $v_3 k$ is NOT an edge of H, as otherwise $(v_3, v_4) \to (v_3, h) \to (k, h)$, a contradiction, since S_{hk} is the non-trivial component that implies (v_4, v_5).

Now we have an exobiclique on the vertices $a, v_2, d, f, v_4, k, v_1, b, c, e, v_3, g, h, j$. Note that every vertex in $\{v_1, b, c, e\}$ is adjacent to every vertex in $\{v_3, g, h, j\}$. $\diamond$

Theorem 5.1 implies the correctness of Step 2. Specifically, we have the following Corollary.

Corollary 5.2 *If within Step 2 of the algorithm, we encounter a non-trivial component S such that we cannot add either S^* or S'^* to the current D, then H has an exobiclique.*

Proof: We cannot add S^* and $(S')^*$ because the additions create circuits in $D \cup S^*$ respectively $D \cup (S')^*$. If either circuit leads to (iii) (in Theorem 5.1) we are done by Theorem 3.1. If both lead to (i) or (ii) (in Theorem 5.1), we proceed as follows. Assume $(x_0, x_1), \ldots, (x_n, x_0)$ is a shortest circuit created by adding S^* to the current D, and $(y_0, y_1), \ldots, (y_m, y_0)$ is a shortest circuit created by adding S'^* to the current D. We may assume that S^* contributes (x_n, x_0) to the first circuit and S'^* contributes (y_m, y_0) to the second circuit. Note that S^* and S'^* do not contribute other pairs to these circuits, as this would contradict (i). Indeed, if say pairs $(x_n, x_0), (x_i, x_{i+1})$ are in the same component of H^+, then x_n, x_i or x_0, x_{i+1} are in the same H_a by Remark 3.1.

We assume each $x_i \in H_{a_i}$ and $y_j \in H_{b_j}$, thus all pairs (x_i, x_{i+1}), (y_j, y_{j+1}) are in non-trivial components (not implied by non-trivial components). Thus S must contain both (x_n, x_0) and (y_0, y_m). If Z is empty, we can conclude by Remark 3.1 that $a_n = b_0$ and $a_0 = b_m$, and therefore (x_{n-1}, y_0), (x_{n-1}, x_n) are in the same non-trivial component, and (y_{m-1}, y_m), (y_{m-1}, x_0) are also in the same non-trivial component, and hence (x_{n-1}, y_0), (y_{m-1}, x_0) are already in D. Therefore

$$(x_0, x_1), (x_1, x_2), \ldots, (x_{n-1}, y_0), (y_0, y_1), \ldots, (y_{m-2}, y_{m-1}), (y_{m-1}, x_0)$$

is a circuit in D, contrary to assumption. If Z is non-empty, we can proceed in exactly the same manner, knowing that no vertex x_i or y_j lies in H_1. $\diamond$

6 Structure of a circuit at Step 3

We consider what happens when a circuit is formed during the execution of Step 3 of our algorithm. In what follows we specify the length and the properties of a circuit in D, considering the level by level construction of envelope of D, $\widehat{D}$. By *minimal chain* between x_0, x_n we mean the following :

1. first time (the smallest level) that there is a sequence $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n)$ of the pairs in D (when we compute $\widehat{D}$) implying (x_0, x_n) in D.
2. None of the pairs (x_i, x_{i+1}) is by transitivity.

When we consider the $T(x, y)$ we note the following:
If (x, y) is by a minimal chain $(x, x_1), (x_1, x_2), \ldots, (x_{n-1}, y)$ then (x, y) is NOT implied by some pair (x', y') for which the length of the minimal chain between (x', y') is less than n.

The *minimal circuit* C is the first time created circuit during computation of $\widehat{D}$ and it has the minimum length. None of the pairs of the circuit is by transitivity. Each pair is an original pair.

Lemma 6.1 *Let (x, y) be a pair in D after step 2 of the algorithm, and current D has no circuit. If (x, y) is obtained by a minimal chain $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_{n+1})$; $x_0 = x$ and $x_{n+1} = y$ then*

1. *x_i, x_{i+2} have always different colors.*
2. *If x, y have the same color then $n \leq 3$ and x_n, y have different colors.*
3. *If x, y have different colors then $n \leq 2$.*
   - *If $n = 2$ then x_n, y have the same color.*
   - *If $n = 1$ and xy is not an edge then x, x_1 have the same color*
   - *If $n = 1$ and xy is an edge then x_1, y have the same color.*

Proof of 1 Suppose first all three x_i, x_{i+1}, x_{i+2} have the same color, say black. Recall that a pair, such as (x_i, x_{i+1}), is only chosen inside a non-trivial component S, or in the envelope of D. Since our (x_i, x_{i+1}) is not by transitivity, in either case there exists a white vertex a of H such that the pair $(x_i, a) \in D$ dominates (x_i, x_{i+1}) in H^+, i.e., a is adjacent in H to x_{i+1} but not to x_i. For a similar reason, there exists a white vertex b of H adjacent to x_{i+1} but not to x_i, i.e., the pair $(x_{i+1}, b) \in D$ dominates (x_{i+1}, x_{i+2}) in H^+.

We now argue that a is not adjacent to x_{i+2}: otherwise, $(x_i, a) \in D$ also dominates the pair (x_i, x_{i+2}) and hence (x_i, x_{i+2}) is also in D at the same level as (x_i, x_{i+1}), contradicting the minimality of our chain.

Next we observe that the pair (x_i, a) is not by transitivity. Otherwise $(x_i, x_{i+1}), (x_{i+1}, x_{i+2})$ can be replaced by a chain obtained from the pairs that imply (x_i, a) together with the pair (a, x_{i+2}). The pair (a, x_{i+2}) lies in the same giant component of H^+ as $(x_i, x_{i+2}) \in D$ since the edges $x_{i+1}a, x_{i+2}b$ are independent. Since all pairs of a non-trivial component are chosen or not chosen for D at the same time, this contradicts the minimality of the circuit. Thus (x_i, a) is dominated in H^+ by some pair $(c, a) \in D$. Since x_i, a have different colors, this means c is a white vertex adjacent to x_i. Note that c is not adjacent to x_{i+2}, since otherwise $(c, a) \in D$ dominates (x_{i+2}, a), which would place (x_{i+2}, a) in D, contrary to $(a, x_{i+2}) \in D$.

Now we claim that b is not adjacent to x_i in H: else the pair $(x_{i+1}, b) \in D$ dominates in H^+ the pair (x_{i+1}, x_i), while $(x_i, x_{i+1}) \in D$. Finally, c is not adjacent to x_{i+1}. Otherwise, cx_{i+1}, bx_{i+2} are independent edges in H, and cx_i, bx_{i+2} are also independent edges in H, and therefore the pairs (x_i, x_{i+2}) and (x_{i+1}, x_{i+2}) are in the same non-trivial component, contradicting again the minimality of our chain. Now (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}), (x_i, x_{i+2}) are in non-trivial components. Since there is no circuit in D, according to the rules of the algorithm $(x_i, x_{i+2}) \in D$, contradicting the minimality of the chain.

We now consider the case when x_i, x_{i+2} are black and x_{i+1} is white. As before, there must exist a white vertex a and a black vertex b such that the pair (a, x_{i+1}) dominates (x_i, x_{i+1}) and the pair (b, x_{i+2}) dominates (x_{i+1}, x_{i+2}); thus ax_i is an edge of H and so is bx_{i+1}. Note that the pair (a, x_{i+1}) dominates the pair (x_i, x_{i+1}) which dominates the pair (x_i, b). Therefore we can replace x_{i+1} by b and obtain a chain which is also minimal. Now (b, x_{i+2}) is by transitivity and we can replace it by a minimal chain. This would contradict the minimality of the chain.

We show that $n \leq 4$.

Set $x_0 = x$ and $x_{n+1} = y$. Let i be the minimum number such that x_i, x_{i+1} have the same color, say black and x_{i+2}, x_{i+3} are white. Let x' be a vertex such that $(x_i, x') \in D$ dominates (x_i, x_{i+1}). Note that if x_{i+4} exists then it is black. If x_{i+4} exists and $n \geq 5$ then x_{i+4} is white, and $x'x_{i+4}$ is not an edge as otherwise (x_i, x') would dominates (x_i, x_{i+4}) and we get a shorter chain. Now let y' be a vertex such that $(x_{i+4}, y') \in D$ dominates (x_{i+4}, x_{i+5}). Now $y'x_{i+1}$ is not an edge as otherwise (x_{i+4}, y') would dominates (x_{i+4}, x_{i+1}) and we get circuit $(x_{i+1}, x_{i+2}), (x_{i+2}, x_{i+3}), (x_{i+3}, x_{i+4}, (x_{i+4}, x_{i+1})$. Now $x'x_{i+1}, y'x_{i+4}$ are independent edges and hence (x_{i+1}, x_{i+4}) is in a non-trivial component. Note that each non-trivial component or its coupled is in D. (x_{i+4}, x_{i+1}) is not in D as otherwise we get a circuit in D, and hence $(x_{i+1}, x_{i+4}) \in D$, and we get a shorter chain. Thus we may assume that x_{i+4} does not exist. This means $x_{i+4} = y$. Now by minimality assumption for i, $x_{i-1} = x_0$ and hence $n \leq 4$.

Proof of 2. Suppose x, y have the same color. If $n = 4$ then according to (1) x, x_1, x_4, y have the same color opposite to the color of x_2, x_3. In this case $i = 0$, and $x_1x', y'y$ are independent edges and hence (x_1, y) is in a non-trivial component and hence $(x_1, y) \in D$, contradicting the minimality of the chain. Therefore $n \leq 3$. Now we show that in this case if $n = 3$ then x_3, y have different colors. In contrary suppose x_3, y have the same color. According to (1), x_1, x_2 have the same color opposite to the color of x, y, x_3. Recall that $i = 1$; (x_1, x') dominates (x_1, x_2). Let y'' be a vertex such that (x_3, y'') dominates (x_3, y). $y''x$ is not an edge as otherwise (x_3, y'') would dominate (x_3, x) and we get a circuit. Let x'' be a vertex such that $(x'', x_1) \in D$ dominates (x, x_1). Now $x'x''$ is not an edge as otherwise (x_1, x') dominates (x'', x_1) and we get a circuit in D. Now x_2x is an edge of H as otherwise x_2x', xx'' are independent edges and hence (x, x_2) would be in a non-trivial component that has already been placed in D, contradicting the minimality of the chain. Now $(x_2, x_3), (x_3, y')$ would imply (x_2, y') and $(x_2, y') \rightarrow (x, y') \rightarrow (x, y)$. This would be a contradiction to the minimality of the chain. In fact we obtain (x, y) in less number of steps of transitivity.

Proof of 3. Suppose x, y have different colors. If $n = 4$ then according to (1) x, x_3, x_4 have the same color and opposite to the color of x_1, x_2, y. Note that here $i = 1$. We observe that xy is not an edge as otherwise (x_4, y) would dominates (x_4, x) and hence we get a circuit in D. Let x'' be a vertex such that $(x'', x_1) \in D$ dominates (x, x_1). Now $x'x''$ is not an edge as otherwise (x_1, x') dominates (x'', x_1) and we get a circuit in D. Now x_2x is an edge of H as otherwise x_2x', xx'' are independent edges and hence (x, x_2) would be in a non-trivial component that has already been placed in D, contradicting the minimality of the chain. Now $(x_2, x_3), (x_3, x_4), (x_4, y)$ would imply (x_2, y) and (x_2, y) dominates (x, y). This would be a contradiction to the minimality of the chain. In fact we obtain (x, y) in fewer number of transitivity application. Therefore $n \leq 3$. Now it is not difficult to see that either $n = 2$ and x, x_1 have the same color opposite to the color of x_2, y or $n = 1$.

If $n = 1$ and x_1 have the same color as x then clearly xy is not an edge. So we may assume xy is not an edge. Now $x''y'$ is not an edge and hence yy', xx'' are independent edges. This implies that (x, y) is in a non-trivial component, contradicting the minimality of the chain. $\diamond$

Corollary 6.2 *Let (x, y) be a pair in D after step 2 of the algorithm, and current D has no circuit.*

- *Suppose x,y have the same color and (x,y) is implied by pair (x,w) such that (x,w) is by transitivity with a minimal chain $(x,x_1),(x_1,x_2),\ldots,(x_m,w)$. Then $m=2$ and x,x_1 have the same color and opposite to the color of x_2,w.*

- *Suppose x,y have different colors and (x,y) is implied by pair (w,y) such that (w,y) is not in a non-trivial component. Then (w,y) is by transitivity with a minimal chain $(w,w_1),(w_1,w_2),(w_2,y)$ where w_1,w_2 have the same color opposite to the color of w,y.*

Proof: If x,y have the same color then by Lemma 6.1 we have $m=2$ or $m=1$. If $m=2$ then x,x_1 have the same color and opposite to the color of x_2,w. When $m=1$ then by Lemma 6.1 (5), w_1,y have the same color. Note that (w_1,w) dominates (w_1,y) and (w_1,y) is in D at the same time (w_1,w) placed in D. Therefore we use the chain $(x,w_1),(w_1,y)$ in order to obtain (x,y), contradiction. If x,y have different colors then by Lemma 6.1 either $m=2$ or $m=3$. If $m=3$ then w,w_1,y have the same color and opposite to the color of w_2,w_3. Let w' be a vertex such that $(w,w')\in D$ dominates (w,w_1). We observe that w_1,x is not an edge as otherwise (w_1,y) implies (x,y) and hence we obtain (x,y) in an earlier level or in fewer step of transitivity application since $(w_1,w_2),(w_2,w_3),(w_3,y)$ are in D. Now wx,w_1w' are independent edges and hence (x,w_1) is already in D, so we may use the chain $C=(x,w_1),(w_1,w_2),(w_2,w_3),(w_3,y)$. Now by considering the chain C we would obtain (x,y) in some earlier step since w_1,w_2 have different colors, and this is a contradiction by Lemma 6.1 (1). Therefore $n=2$. $\diamond$

Now by Lemma 6.1 and Corollary 6.2 we have the following.

Corollary 6.3 *Let $C=(x_0,x_1),(x_1,x_2),...,(x_{n-1},x_n),(x_n,x_0)$ be a minimal circuit, formed at Steps 3 of the Algorithm. Then $n=3$ and x_0,x_3 have the same color and opposite to the color of x_1,x_2.*

Proof: We may assume that non of the pair (x_i,x_{i+1}) in C is by transitivity as otherwise we replace (x_i,x_{i+1}) by a minimal chain between x_i,x_{i+1}. Now we just need to apply Lemma 6.1 and Corollary 6.2. $\diamond$

By Corollary 6.3 we may assume the length of every minimal circuit is of the following form.

$C=(x_0,x_1),(x_1,x_2),(x_2,x_3),(x_3,x_0)$, x_0,x_3 *are white*, x_1,x_2 *are black vertices.*

We say a pair (x,y) is *simple* if $(x,y)\in S*$ for some non-trivial component S otherwise (x,y) is called a *complex pair* .

Lemma 6.4 *If we encounter a minimal circuit $C=(x_0,x_1),(x_1,x_2),\ldots,(x_3,x_0)$ at Step 3 then there is a non-trivial component S such that the envelope of every complete set D_1 with $S\subset D_1$ contains a circuit.*

(By keeping S in D at step (2) of the algorithm and replacing some of the giant components $R\neq S,S'$ by R' at step 2 we still encounter in $\widehat{D}$.)

Proof: Note that x_0,x_3 are white and x_1,x_2 are black vertices. Note that according to the definition of minimal circuit none of the (x_i,x_{i+1}) is by transitivity. Therefore there is pair (x_1,w)

in D that dominates (implies) (x_1, x_2) and (u, x_3) is a pair in D that dominates (implies) (x_2, x_3), and there is a vertex v such that (x_3, v) dominates (implies) (x_3, x_0) and there is a vertex z such that $(z, x_1) \in D$ dominates (x_0, x_1). Note that by Lemma 2.5 if (x_2, x_3) is a simple pair then (x_2, x_3) is in a non-trivial component, and by Lemma 2.6 if (x_1, x_2) is a simple pair then it is implied by a non-trivial component $S_{x_1 w}$. We also note that there could be some other w' such that (x_1, w') implies (x_1, x_2) but the argument would be the same and in fact $Dict(x_1, w)$ would be the same as $Dict(x_1, w')$.

Before we continue we observe that x_0x_2 is an edge of H.

For contrary suppose x_0x_2 is an edge. Let (p, x_1) be a pair in D that dominates (x_0, x_1) ((x_0, x_1) is not by transitivity). Now wp is not an edge as otherwise (x_1, w) would dominates (x_1, p) implying an earlier circuit in D. Now px_0, wx_2 are independent edges and hence (x_0, x_2) would be in a non-trivial component and consequently (x_0, x_2) has been already placed in D (if (x_2, x_0) is in D then we would have an earlier circuit) implying a shorter circuit. Therefore x_0x_2 is an edge.

Remark : In what follows we decompose each of the pairs $(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$, meaning that we analyze the steps in computing $\widehat{D}$ to see how we get these pairs. Each pair is either a simple pair or is a complex pair. If (x_i, x_{i+1}) is a simple pair then there is a non-trivial component S_i such that either $(x_i, x_{i+1}) \in S_i$ or (x_i, x_{i+1}) is implied by S_i. If pair (x_i, x_{i+1}) is a complex pair then after decomposing (x_i, x_{i+1}) we obtain a non-trivial component S_i (The description of obtaining S_i is explained below). We show the relationships between S_i and S_j for every $0 \le i, j \le 3$. We show that if (x_i, x_{i+1}) is a complex pair and (x_j, x_{j+1}) is also a complex pair then $S_i = S_j$ and S_j is the dictator component. We show that if (x_1, x_2) is a complex pair and (x_0, x_1) is in a non-trivial component then $(x_0, x_1) \in S_1$ and S_1 is the dictator component. All the relationships are investigated in the Claims 6.6, 6.7, 6.9, 6.11. Suppose (x_i, x_{i+1}) is a simple pair either in non-trivial component S_i or implied by a non-trivial component S_i and suppose (x_j, x_{j+1}) is a complex pair for some $0 \le j \le 3$ such that $S_i \ne S_j$. Then Claim 6.12 implies that replacing S_i by S'_i at step 2 and keeping S_j in D at step 2 we still get a circuit in the envelope of D.

Claim 6.5 *If (x_1, x_2) ((x_3, x_0)) is not simple then (x_1, w) ((z, x_3)) is by transitivity.*

Proof: For contrary suppose (x_1, w) is not by transitivity. Thus there is some (w', w) that imply (x_1, w). Now (w', w) is not in a non-trivial component as otherwise (x_1, x_2) is implied by $S_{w'w}$ and hence (x_1, x_2) is simple. Thus (w', w) is by transitivity and by Corollary 6.2 there are w'_1, w'_2 such that $(w', w'_1), (w'_1, w'_2), (w'_2, w)$ are in D and they imply (w', w). Now (x_0, x_1) and (x_0, w') are in D at the same time ((x_0, x_1) implies (x_0, w')). Moreover (w'_2, w) implies (w'_2, x_2), and they are in D at the same time. Now we would have the circuit $(x_0, w'), (w', w'_1), (w'_1, w'_2), (w'_2, x_2), (x_2, x_3), (x_3, x_0)$, contradicting the minimality of the original circuit. $\diamond$

In the rest of the proof we often use similar argument in the Claim 6.5 and we do not repeat it again. In what follows we consider the decomposition of each of the pairs (x_1, w) , (u, x_3), and (x_3, v).

Decomposition of (x_1, w)

Suppose (x_1, x_2) is not simple. Then by Claim 6.5 (x_1, w) is by transitivity and hence by Lemma 6.1 there are vertices w_1^1, w_2^1, w^1, $w^1 = w$ such that the chain $(x_1, w_1^1), (w_1^1, w_2^1), (w_2^1, w^1)$ is minimal and imply (x_1, w). Moreover w_1^1, w^1 are white and x_1, w_1^1 are black, and none of the $w_1^1 w_2^1, x_1 w^1$ is an edge of H. In general suppose (x_1, w^1) is obtained after m steps; meaning that (x_1, w_1^1) is not simple and is obtained after $m-1$ steps of implications (transitivity and implication).

To summarize: for every $1 \leq i \leq m-1$ we have the following.

1. By similar argument in Claim 6.5 (x_1, w^i) is by transitivity and hence there are vertices w^i, w_1^i, w_2^i such that $(x_1, w_1^i), (w_1^i, w_2^i), (w_2^i, w^i)$ is a minimal chain and imply (x_1, w^i)
2. w^i, w_2^i are white and w_1^i is black
3. (x_1, w^{i+1}) implies (x_1, w_1^i).
4. none of the $w_1^i w_2^i, x_1 w^i, x_1 w_2^i, w_2^i x_2$ is an edge of H.

Since m is minimum we have the following :

1. There is no edge from w^{i+1} to w_1^{i-1} as otherwise (x_1, w^{i+1}) would dominate (x_1, w_1^{i-1}) and hence we get a shorter chain $(x_1, w_1^{i-1}), (w_1^{i-1}, w_2^{i-1}), (w_2^{i-1}, w^{i-1})$, and consequently we obtain (x_1, w^1) in less than m steps.
2. There are vertices f^i, $1 \leq i \leq m-1$ such that (w_2^i, f^i) implies (w_2^i, w^i).
3. $f^i w^j$, $j \geq i+1$ is not an edge of H, and $f^i w_2^j$, $j \geq i+2$ is not an edge, as otherwise (w_2^i, f^i) would dominates (w_2^i, w^j) and hence we use the chain $(x_1, w_2^i), (w_2^i, w^j)$ to obtain (x, w^1) in less than m steps.
4. $w_2^i x_1$ is not an edge as otherwise (w_1^i, w_2^i) would imply (w_1^i, x_1) and hence we get an earlier circuit because (x_1, w_1^i) is in D.
5. $w_1^i w^i$ is an edge as otherwise $w_1^i w^{i+1}, w^i w^{i-1}$ are independent edges and hence (w^{i+1}, w_1^{i-1}) is in a non-trivial component already placed in D (otherwise we would have an earlier circuit using $(x_1, w^{i+1}), (w^{i+1}, w_1^{i-1}), (w_1^{i-1}, w_2^{i-1})....)$), a contradiction to the minimality of (x_1, w^1).
6. There are vertices a, b such that $x_1 a$ and $w^m b$ are independent edges; (x_1, w^m) is in a non-trivial component S_1.

Since (x_1, w^m) is simple and x_1, w^m have different colors, there are vertices a, b such that $x_1 a, w^m b$ are independent edges. Note that $w^m w_1^j$ is not an edge for $j < m$ as otherwise (x_1, w^m) would imply $(x_1, w_1^j) \in D$ and hence we get a shorter chain $(x_1, w_1^j), (w_1^j, w_2^j), (w_2^j, w^j)$, and we get (x_1, w) in less than m steps.

Since $w^{m-1} x_1$ is not an edge, $a w_1^{m-2}$ and $a f^{m-2}$ are edges of H as otherwise (x_1, w^{m-2}) is in a non-trivial component that has already placed in D (since we are in step 3, and (w^{m-2}, x_1) is not in D as it would yield an earlier circuit) implying (x_1, w) in less than m steps.

We show that $f^i w^{i+1}$ is an edge. Otherwise $(w_1^i, w_2^i), (w_2^i, f^i)$ imply $(w_1^i, f^i) \in D$ and now $(w_1^i, f^i) \to (w^{i+1}, f^i)$ and hence $(w^{i+1}, w^i) \in D$. This would contradict the minimality of the chain, fewer number of steps in obtaining (x_1, w^1).

Now observe that $(a, w^m) \to (w_1^{m-2}, w^m) \to (w_1^{m-2}, f^{m-1})$. Also $(w_1^{m-2}, f^{m-1}) \to (w^{m-1}, w^{m-2})$. By continuing this we see that $(w^{i-1}, w^i) \to (w_1^{i-2}, f^{i-1}) \to (w^{i-2}, w^{i-1})$, $3 \le i \le m-1$. Finally we see that (x_1, b) and (x_2, f^1) are in the same non-trivial component.

Note that $f^1 u$ is not an edge as otherwise (w_2^1, f^1) would dominate (w_2^1, u) and we get an earlier circuit $(x_1, w_1^1), (w_1^1, w_2^1), (w_2^1, u), (u, x_3)$. Therefore $(x_2, f^1) \to (u, w^1)$ since there is no edge from f^1 to u.

Now $(x_2, f^1), (x_1, w^m) \in S_1$.

Claim 6.6 *Suppose* (x_1, x_2) *is a complex pair and* (x_0, x_1) *is in a non-trivial component* S_0*. Then* $(x_0, x_1) \in S_1$*.*

Proof: Since x_0, x_1 have different colors, there are vertices p, q such that $x_0 p, x_1 q$ are independent edges. Observe that $x_0 f^{m-1}$ is not an edge as otherwise (w_2^{m-1}, f^{m-1}) dominates (w_2^{m-1}, x_0) and hence we get an earlier circuit $(x_0, x_1), (x_1, w_1^{m-1}), (w_1^{m-1}, w_2^{m-1}), (w_2^{m-1}, x_0)$.

Note that pw^1 is not an edge as otherwise (p, x_1) dominates (w^1, x_1) while (x_1, w^1) is in D. Recall that $x_0 x_2$ is an edge of H. Observe that qx_2, ax_2 are edges of H as otherwise (x_1, x_2) would be in a non-trivial component.

Also qf^{m-1}, af^{m-1} are both edges of H as otherwise $x_1 q, f^{m-1} w^{m-1}$ are independent edges and hence (x_1, w^{m-1}) is in a non-trivial component, and we obtain (x_1, x_2) in less than m steps.

Note that if ap is an edge and qb is an edge then (x_0, x_1) and (x_1, w^m) are in a same non-trivial component, and claim is proved. Therefore we may assume at least one of the qb, ap is not an edge of H. We prove the claim for qb not being an edge of H and the proof for $ap \notin E(H)$ is similar. When qb is not an edge of H we need to see that $x_0 x_2, qx_2$ are edges of H while $w^m x_2$ is not an edge of H and $f^{m-1} w^m, f^{m-1} q$ are edges of H while $x_0 f^{m-1}$ is not an edge. These would imply that (x_0, x_1) and $(x_1, w^m), (q, w^m)$ are in a same non-trivial component S_1. $\diamond$

Decomposition of (u, x_3)

If (x_2, x_3) is a complex pair then (u, x_3) is obtained after $n > 1$ steps as follows.

There are vertices $u^1 = u$ and g^i, u^i, u_1^i, u_2^i, $1 \le i \le n$, such that

1. (u^i, x_3) implies (u_2^{i-1}, x_3), $i \ge 2$.
2. $(u^i, u_1^i), (u_1^i, u_2^i), (u_2^i, x_3)$ imply (u^i, x_3). u^i is white and u_1^i, u_2^i are black.
3. for $1 < i \le n$, $u^i u_2^{i-1}$ is an edge.
4. (u_2^{n-1}, x_3) is in a non-trivial component.
5. (u_1^i, g^{i+1}) is a pair in D that dominates (u_1^i, u_2^i) for $1 < i \le n-1$.
6. There is a vertex c such that $u^n u_2^{n-1}, cx_3$ are independent edges of H, and (u_2^{n-1}, x_3), (u^n, x_3) are in a non-trivial component S_2.

Since (u_2^{n-1}, x_3) is simple and u_2^{n-1}, x_3 have different colors, (u_2^{n-1}, x_3) is in a non-trivial component S_2 and then there are vertices u^n, c such that $u^n u_2^{n-1}, cx_3$ are independent edges of H, and hence $(u_2^{n-1}, x_3), (u^n, x_3)$ are in the non-trivial component S_2. Note that u^i, u_2^j, $j \leq i-2$ are not adjacent as otherwise $(u^i, x_3) \rightarrow (u_2^j, x_3)$ and hence we get (u, x_3) in less than n steps. Also $u^i u_2^i$ is an edge as otherwise $u^i u_1^{i-1}, u^{i+1} u_2^i$ are independent edges and hence (u^{i-1}, u^i) is in a non-trivial component placed in D, implying an earlier (shorter) chain. Note that by definition $(u_1^i, g^{i+1}) \rightarrow (u_1^i, u_2^i)$. Observe that $w^1 u_2^i$ is not an edge as otherwise $(x_1, w^1) \rightarrow (x_1, u_2^i)$ and this contradicts the minimality of circuit. Observe that $u^{i+1} u^{i-1}$ is not an edge as otherwise (u^{i+1}, x_3) would implies (u_2^{i-1}, x_3) and hence we would obtain (u, x_3) in less than n steps. By the same reason $x_2 u^2$ is not an edge. Note that $g^i u_2^i$ is not an edge as otherwise $(u_1^{i-1}, g^i) \rightarrow (u_1^{i-1}, u_2^i)$ and hence we obtain (u, x_3) in less than n steps. Moreover $u_1^{i-1} g^{i+1}$ is not an edge as otherwise $(u^i, u_1^i), (u_1^i, g^{i+1})$ would imply $(u^i, g^{i+1}) \in D$. As a consequence $(u^i, g^{i+1}) \rightarrow (u_2^{i-1}, g^{i+1}) \rightarrow (u_2^{i-1}, u_2^i) \in D$ and therefore we obtain (u, x_3) in less than n steps. By applying similar argument we conclude $x_2 g^2$ is an edge.

Now we have that $(u^1, w^1) \rightarrow (u_2^1, w^1) \rightarrow (u_2^1, x_2) \rightarrow (u^2, x_2) \rightarrow (u^2, g^2)$. For every $2 \leq i \leq n-2$, we have $(u^i, g^i) \rightarrow (u_2^i, g^i) \rightarrow (u_2^i, u_1^{i-1}) \rightarrow (u^{i+1}, u_1^{i-1}) \rightarrow (u^{i+1}, g^{i+1})$. Finally we have $g^{n-1} c$ is an edge as otherwise $u_2^{n-2} g^{n-1}, cx_3$ are independent edges an hence (u_2^{n-2}, x_3) is in a non-trivial component and (u, x_3) is obtained in less than n steps.

Now $(u^{n-1}, g^{n-1}) \rightarrow (u_2^{n-1}, g^{n-1}) \rightarrow (u_2^{n-1}, c)$

Claim 6.7 *If (x_2, x_3) is a complex pair and (x_3, x_0) is a simple pair implied by non-trivial component S_3 then $S_3 = S_2$.*

Proof: Since (x_3, x_0) is implied by (x_3, v) and (x_3, x_0) is simple, (x_3, v) is in a non-trivial component and there are independent edges $x_3 c, vd$ of H. Note that $u^{n-1} v$ is not an edge as otherwise (x_3, v) would imply (x_3, u^{n-2}) while $(u^{n-2}, x_3) \in D$. However cu^{n-1} is an edge as otherwise $u^{n-1} u_2^{n-2}, x_3 c$ are independent edges and hence (x_2^{n-2}, x_3) is in D, a contradiction. Now $u_2^{n-1} u^{n-1}, cu^{n-1}$ are edges of H. We note that $w^1 c$ is an edge as otherwise $x_2 w^1, x_3 c$ are independent edges and hence (x_2, x_3) would be in a non-trivial component. We show that $w^1 v$ is an edge as otherwise $(d, v) \rightarrow (w^1, v) \rightarrow (w^1, x_0)$ and hence $(w^1, x_0) \in D$ while $(x_0, x_1), (x_1, w^1)$ are also in D, yielding an earlier circuit in D. Recall that $w^1 u_2^{n-1}$ is not an edge. Now $cu^{n-1}, u_2^{n-1} u^{n-1}$ are edges of H while vu^{n-1} is not an edge and $w^1 c, w^1 v$ are edges of H while $w^1 u_2^{n-1}$ is not an edge. These imply that (x_3, v) and (u^n, x_3) are in a same non-trivial component S_2.

Claim 6.8 *Suppose for some $1 \leq i \leq n$, (u^i, u_1^i) is a simple pair inside non-trivial component R_1 and (u_1^i, u_2^i) is a simple pair implied by a non-trivial component R_2. Then for any selection R_3 from $\{R_1, R_1'\}$ instead of R_1 and any selection R_4 from $\{R_2, R_2'\}$ instead of R_2 at step (2); the pair (u_2^{i-1}, x_3) is in D, and hence the complex pair (x_2, x_3) is in D.*

Proof: Note that since $u^i u_2^i$ is an edge, (u_1^i, u_2^i) is implied by a non-trivial component. Let $u^i a_i, u_1^i b_i$ be the independent edges and $u_1^i c_i, d_i e_i$ be independent edges that (u_1^i, e_i) implies (u_1^i, u_2^i). Note that $u_2^i e_i$ and $u_2^i c_i, u_2^i b_i$ are edges of H. Note that (u^i, u_1^i) implies (u^i, c_i) and (c_i, d_i) is in a non-trivial component. Thus $d_i u^i$ is not an edge as otherwise (c_i, d_i) dominates (c_i, u^i) and we get a shorter circuit. Similarly $a_i e_i$ is not an edge as otherwise (u_1^i, e_i) dominates (u_1^i, a_i)

a contradiction. Now $e_i u_2^{i-1}$ is an edge as otherwise $u^i u_2^{i-1}, e_i d_i$ are independent edges and since (u^i, e_i) is in D (all the non-trivial components have been added) , (u_2^{i-1}, e_i) implies (u_2^{i-1}, u_2^i) and we obtain (u^1, x_3) in less than n steps. Also $u_2^{i-1} b_i, u_2^{i-1} c_i$ are edges of H as otherwise $u_2^{i-1} u^i, u_1^i b_i$ or $u_2^{i-1} u^i, u_1^i c_i$ are independent edges and hence (u_2^{i-1}, u_1^i) is in a non-trivial component and we obtain (u^1, x_3) in less than n steps.

Now this would imply that no matter what the algorithm selects from one of the $S_{u^i u_1^i}, S_{u_1^i u^i}$ at step (2) and no matter what the algorithm selects from one of the $S_{u_1^i e_i}, S_{e_i u_1^i}$ at step (2), one of the pair (u^i, u_2^i), (e_i, u_2^i), and (c_i, u_2^i) appears in $\widehat{D}$.

Suppose we should have selected $S_{e_i u_1^i}$ and $S_{u_1^i u^i}$ at step (2). Now (u_1^i, u^i) dominates (u_1^i, u_2^i) and hence we have (e_i, u_2^i). Thus $(e_i, x_3) \in D$ which implies $(u_1^{i-1}, x_3) \in D$. This means that instead of pair (u^i, x_3) we would have (e_i, x_3) and we would apply the same decomposition for (e_i, x_3) as decomposition of (u^i, x_3). If we should have selected (c_i, d_i) and (d_i, a_i) then (d_i, u^i) would dominate (d_i, u_2^i) and hence (c_i, u_2^i) would be in D implying that $(c_i, x_3) \in D$ which would imply $(u^{i-1}, x_3) \in D$. The similar argument is implied for other selections of R_3, R_4. $\diamond$

Claim 6.9 *If (x_1, x_2) is a complex pair and (x_2, x_3) is also a complex pair then $S_1 = S_2$.*

Proof: We need to see that there is a direct path from (x_1, w^m) to (u^1, w^1). Moreover there is a directed path from (u^1, w^1) to (u_1^1, x_2). There is also a direct path from (u_2^1, x_2) to (u^n, x_3). We need to observe that $(u_2^1, x_2) \in S_1$ and $(u^n, x_3) \in S_2$ and since there is a direct path from S_1 to S_2, $S_1 = S_2$. $\diamond$

Decomposition of (x_3, v)

Suppose (x_3, x_0) is a complex pair and it is obtained after t steps. This means there are vertices v^i, v_1^i, v_2^i for $1 \le i \le t$ and $v^1 = v$ such that :

1. (x_3, v^{i+1}) implies (x_3, v_1^i)
2. $(x_3, v_1^i), (v_1^i, v_2^i), (v_2^i, v^i)$ imply (x_3, v^i). v^i, v_2^i are black and v_1^i is white.
3. $v^i v_2^{i-1}$, $2 \le i \le t$ is an edge.
4. (x_3, v^t) is in a non-trivial component, and v^t is black.

There are vertices d, e such that $x_3 d, v^t e$ are independent edges and $v^t v_1^{t-1}$ is an edge. Let $S_3 = S_{ex_3}$. Note that dv_1^{t-1} is also an edge. Let g^{t-1} be a vertex that (v_2^{t-1}, g^{t-1}) implies (v_2^{t-1}, v^{t-2}). As we argued in the decomposition of (x_1, w^1), $g^{t-1} v^t$ is an edge of H. We nota that dg^{t-1} is an edge as otherwise since $x_3 v^{t-1}$ is not an edge, $x_3 d, v^{t-1} g^{t-1}$ are independent edges and we obtain (x_3, v) in less than t steps. We also note that $v^t x_0$ is not an edge of H.

The proof of the following Claim is analogues to proof of Claim 6.9 however we repeat it for sake of completeness.

Claim 6.10 *If (x_2, x_3) and (x_3, x_0) are complex pairs then $S_2 = S_3$.*

Proof: Observe that $g^{t-1}u_2^{n-1}$ is not an edge as otherwise (v_2^{t-1}, g^{t-1}) would imply (v_2^{t-1}, u_2^{n-1}) and now we have an earlier circuit $(u_2^{n-1}, x_3), (x_3, v_1^{t-1}), (v_1^{t-1}, v_2^{t-1}), (v_1^{t-1}, u_2^{n-1})$. Recall that $u^{n-1}c$ is an edge. Now $v^t u^{n-1}$ is not an edge as otherwise (x_3, v^t) would imply $(x_3, u^{n-1}) \in D$ while we had $(u^{n-1}, x_3) \in D$ and we have an earlier circuit. Now both u_2^{n-1}, c are adjacent to u^{n-1} and v^t is not adjacent to u^{n-1} and d, v^t both are adjacent to g^{t-1} while u_2^{n-1} is not adjacent to g^{t-1}. Therefore (u_2^{n-1}, x_3) and (x_3, v^t) are in a same non-trivial component. $\diamond$

Claim 6.11 *If (x_1, x_2) and (x_3, x_0) are complex pairs and (x_0, x_1), (x_2, x_3) are simple pairs then $S_1 = S_3$ and $(x_2, x_3), (x_0, x_1) \in S_1$.*

Proof: Note that by Claim 6.6 we have $(x_0, x_1) \in S_1$. The proof of $(x_2, x_3) \in S_3$ is analogues to proof of the Claim 6.6 however for sake of completeness we give the proof. Recall that x_0p, x_1q be the independent edges of H. Note that $g^{t-1}x_2$ is not an edge as otherwise (v_2^{t-1}, g^{t-1}) would dominates (v_2^{t-1}, x_2) and hence we have an earlier circuit $(x_2, x_3), (x_3, v_1^{t-1})(v_1^{t-1}, v_2^{t-1}), (v_2^{t-1}, x_2)$. Also dg^{t-1} is an edge as otherwise $x_3d, g^{t-1}v^{t-1}$ are independent edges and hence (x_3, v^{t-1}) would be in D, and we obtain (x_3, x_0) in less than t steps. Now x_2x_0, dx_0, cx_0 are edges of H while $v^t x_0$ is not an edge of H and $dg^{t-1}, cg^{t-1}, v^t g^{t-1}$ are edges of H while $x_2 g^{t-1}$ is not an edge and hence $(x_2, x_3), (x_3, v^t)$ are in the same non-trivial component.

Now it remains to show $S_1 = S_3$. Recall that $f^{m-1}w^m$ is an edge of H, also $f^{m-1}a$ and $f^{m-1}q$ are edges of H as otherwise $f^{m-1}w^{m-1}, x_1q$ are independent edges and $f^{m-1}w^{m-1}, x_1a$ are independent edges and hence (x_1, w^{m-1}) is in non-trivial component and we obtain (x_1, w^1) in less than m steps. x_3d, ux_2 are independent edges. Note that x_1u is not an edge as otherwise (u, x_3) dominates (x_1, x_3) and hence we get an earlier circuit. Recall that x_2w^m is not an edge. Moreover $v^t v^{t-2}$ is not an edge as otherwise $x_3d, v^t v^{t-2}$ are independent edges and hence (x_3, v^{t-2}) is in a non-trivial component and we get (x_3, v^1) in less than t steps.

If $w^m v^t$ is an edge of H then $(d, v^t) \to (x_0, v^t) \to (x_0, w^m)$ and hence $S_1 = S_3$. So we may assume that $w^m v^t$ is not an edge. If $v^t q$ is an edge of H then $(d, v^t) \to (x_0, v^t) \to (x_0, q)$ and hence $S_0 = S_3$ and by Claims 6.7 $S_0 = S_1 = S_2 = S_3$. If $w^m d$ is an edge then $(a, w^m) \to (x_2, w^m) \to (x_2, d)$ and hence $S_1 = S_3$. So we may assume $w^m d$ is not an edge.

We conclude that $f^{m-1}g^{t-1}$ is an edge as otherwise $(a, w^m) \to (d, w^m) \to (d, f^{m-1}) \to (g^{t-1}, f^{m-1}) \to (g^{t-1}, q) \to (v^t, q) \to (v^t, d)$, implying that $S_2 = S_3'$ a contradiction.

Now $(a, w^m) \to (x_2, w^m) \to (x_2, f^{m-1}) \to (u, f^{m-1}) \to (u, g^{t-1}) \to (x_2, g^{t-1}) \to (x_2, d)$. This would imply that $S_1 = S_3$.

Remark : The decomposition was for each of the pair $(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$. Now for example consider the complex pair (x_2, x_3) implied by (u, x_3). When we decompose (u, x_3) into pairs $(u^1, u_1^1), (u_1^1, u_2^1), (u_2^1, x_3)$ then we recursively decompose (u_2^1, x_3). By applying the decomposition to each of the $(u^1, u_1^1), (u_1^1, u_2^1)$ we reach to the same conclusion as for the pairs $(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$. In fact the circuit C has four pairs that we can view them as external pairs while the pair (u^1, u_1^1) is an internal pair and the same rule applied for it with respect to the neighboring pair (u_1^1, u_2^1).

Now it remains to consider the case when (x_0, x_1) is a simple pair in a non-trivial component S_0 and (x_1, x_2) is a simple pair implied by non-trivial component S_1, and none of the S_2 and S_3 is in set $\{S_0, S_1\}$. In this case we claim the following.

Claim 6.12 *Suppose (x_0, x_1) is a simple pair in non-trivial component S_0 and (x_1, x_2) is a simple pair implied by non-trivial component S_1 such that none of the S_2 and S_3 is in set $\{S_0, S_1\}$. Then by replacing S_0 with S'_0 in D or by replacing S_1 with S'_1 in D and keeping the non-trivial components S_2, S_3 in D at step 2 of the algorithm we still get a circuit $(y_0, y_1), (y_1, y_2), (x_2, x_3), (x_3, y_0)$ in $\widehat{D}$ (See Figure 3).*

Proof: According to since we keep S_2 in D at step (2) the pair (x_2, x_3) appears in D (envelope of D) at step (3). Since (x_0, x_1) is a simple pair and x_0, x_1 have different colors, there are independent edges x_0p, x_1q. There are independent edges x_1a, wb such that (x_1, w) implies (x_1, x_2). Note that x_2q, x_2x_0, x_2a are edges since (x_1, x_2) is not in a non-trivial component. As we argued before in the correctness of step (2), x_0b, pw are not edges of H. qv is an edge as otherwise $(x_0, q) \rightarrow (v, q) \rightarrow (v, x_2)$ and hence $(v, x_2) \in D$, yielding a shorter (earlier) circuit $(x_2, x_3), (x_3, v), (v, x_2)$ which is a contradiction. Suppose first both qb, ap are edges of H. This implies that $S_{x_0x_1} = S_{x_1w}$ and $(x_0, w) \in S_{x_0x_1}$. We note that wv is an edge as otherwise $(x_0, w) \rightarrow (v, w) \rightarrow (v, x_2)$ and hence $(v, x_2) \in D$, yielding a shorter (earlier) circuit $(x_2, x_3), (x_3, v), (v, x_2)$ which is a contradiction. We conclude that (x_3, v) implies $(x_3, w) \in D$. Now if we choose S' instead of S_1 at step (2) then we would have $(x_1, x_0) \in D$ and $(x_1, x_0) \rightarrow (x_1, x_2) \in D$ and $(b, x_1) \in D$. Now we would have the circuit $(w, x_1), (x_1, x_2), (x_2, x_3), (x_3, w)$. We now assume qb is not an edge. Proof for the case $ap \notin E(H)$ is similar. wv is an edge as otherwise $(q, w) \rightarrow (v, q) \rightarrow (v, x_2)$, and again we get an earlier circuit. Now suppose we would have chosen (w, x_1) instead of (x_1, w) at step (2). Note that (w, p) is in a non-trivial component. Now either we have $S_{wp} \in D$ or $S_{pw} \in D$. We continue by the first case $S_{wp} \in D$ at step (2). We have $(w, p) \in D$ and $(p, q) \in D$ that implies (p, x_2) and hence we would have the circuit $(w, p), (p, x_2), (x_2, x_3), (x_3, w)$. If $S_{pw} \in D$ at step (2) then we have $(x_0, b) \in D$. Furthermore (b, q) dominates (b, x_2) and now $(x_0, b), (b, x_2), (x_2, x_3), (x_3, x_0)$ would be a circuit in D. By symmetry the other choices would yield a circuit in D. $\diamond$

We summarize we have the following statement :
Consider the circuit $C = (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ such that x_0, x_3 have the same color and opposite to the color of x_1, x_2. Suppose S_i is a non-trivial component we obtain after the decomposition of the pair (x_i, x_{i+1}). By Claims 6.6, 6.7, 6.9, 6.11 we have the following :

1. If (x_i, x_{i+1}) is in a complex pair and (x_j, x_{j+1}) is a complex pair then $S_i = S_j$.

2. For $i = 1, 3$, if (x_i, x_{i+1}) is a simple pair in non-trivial component S_i and (x_{i-1}, x_i) is a complex pair then $S_{i-1} = S_i$.

3. For $i = 0, 2$ if (x_{i+1}, x_{i+2}) is implied by non-trivial component S_{i+1} and (x_i, x_{i+1}) is a complex pair then $S_i = S_{i+1}$.

4. Suppose (x_i, x_{i+1}) is in a non-trivial component S_i and (x_{i+1}, x_{i+2}) is implied by a non-trivial component S_{i+1}, $i \in \{0, 2\}$. Moreover suppose none of the S_{i+2}, S_{i+3} is in $\{S_i, S_{i+1}\}$. Then by replacing S_i with S'_i or by replacing S_{i+1} with S'_{i+1} and keeping the giant components S_{i+2}, S_{i+3} in D at step 2 of the algorithm we encounter a circuit at step (3).

$\diamond$

Lemma 6.13 *The algorithm computes the $Dict(x, y)$ correctly.*

Proof: Suppose by adding pair (x,y) into D we close a circuit. By Corollary 6.3 a minimal circuit C has four vertices and we may assume $C=(x_0,x_1),(x_1,x_2),(x_2,x_3),(x_3,x_0)$. W.l.o.g assume that x_0,x_3 are white vertices and x_1,x_2 are black vertices.

Recall that the followings determine the dictatorship of a pair (x,y).

(a) If $(x,y)\in S^*$ for some non-trivial component S then $Dict(x,y)=S$.

(b) If x,y have different colors and (x,y) is implied by some pair (u,y) then $Dict(x,y)=Dict(u,y)$.

(c) If x,y have the same color and (x,y) is implied by some pair (x,w) then $Dict(x,y)=Dict(x,w)$.

(d) If x,y have the same color and (x,y) is by transitivity on $(x,w),(w,y)$ then $Dict(x,y)=Dict(w,y)$.

(e) If x,y have different colors and (x,y) is by transitivity on $(x,w),(w,y)$ then $Dict(x,y)=Dict(x,w)$.

Suppose $(u,x_3)\in D$ is a pair implying (x_2,x_3). According to definition $Dict(u,x_3)=Dict(x_2,x_3)$. Since (u,x_3) is by transitivity, by Lemma 6.2 we have the pairs $(u,u_1),(u_1,u_2),(u_2,x_3)$ in $\widehat{D}$. When we compute $\widehat{D}$, (u,x_3) is appeared in $\widehat{D}$ whenever (u,f) and (f,x_3) appeared in $\widehat{D}$ at some earlier level. According to minimality of the chain (u,x_3) either $f=u_2$ or $f=u_1$. First suppose $f=u_2$. Now according to (d) we have $Dict(x_2,x_3)=Dict(u_2,x_3)$. By induction hypothesis we know that $Dict(u_2,x_3)=S_2$. Recall that S_2 is the component obtained after decomposing of (u,x_3) in Lemma 6.4. Therefore $Dict(x_2,x_3)=Dict(u,x_3)=Dict(u_2,x_3)$. Now consider the case $f=u_1$. According to (d) we have $Dict(u,x_3)=Dict(u_1,x_3)$. In this case by using (e) we $Dict(u_1,x_3)=Dict(u_2,x_3)$ because $(u_1,u_2),(u_2,x_3)$ imply (u_1,x_3) and u_1,x_3 have different colors. Similar argument is implied for pair (x_1,x_2), where x_1,x_2 have the same color. $\diamond$

7 Correctness of Step 3 and 4, and 5

At step (3) if we encounter a circuit C in D then according to Lemma 6.4 there is a non-trivial component S that is a dictator for C. We compute this dictator component S by decomposing the pairs of the circuit as explained in Section 3 and finally according to Lemma 6.13 by using *Dict* function.

It is clear that we should not add S to D as otherwise we won't be able to obtain the desired ordering. Therefore we must take the coupled component of every dictator component of a circuit appeared at the first time we take the envelope of D. Lemmas 7.1 and 7.2 would justify the correctness of Steps (3) and (4).

Lemma 7.1 *If all the non-trivial components $S_{ab},S_{ba},S_{bc},S_{cb},S_{ac},S_{ca}$ are pairwise distinct then none of them is a dictator component.*

Proof: By the assumption of the Lemma H is pre-insect with $Z = \emptyset$. Now as we argued in Section 6 if non-trivial component S is a dictator for a circuit then there has to be pair $(x, y), (y, z), (x, z) \in S$. However according to the structure of pre-insect S_{ab} consists of only the pairs (x, y) that $x \in H_1$ and $y \in H_2$. $\diamond$

Lemma 7.2 *There is no circuit at step (4) of the algorithm. (If for every $S \in \mathcal{DT}$ we add $(S')^*$ into D_1 and for every $R \in D \setminus \mathcal{DT}$ we add R^* into D_1 at step (4) then we do not encounter a circuit.)*

Proof: Suppose we encounter a minimal circuit $(x_0, x_1), (x_1, x_2), ..., (x_{n-1}, x_n), (x_n, x_0)$ with the simple pairs such that some of the pairs (x_i, x_{i+1}) are in some $(S')^*$, $S \in \mathcal{DT}$ ($\mathcal{DT}$ is the set of the dictator components).

We say (x_i, x_{i+1}) is an old if it is in S^* and $S \notin \mathcal{DT}$ otherwise (x_i, x_{i+1}) is called a new pair. First suppose that both $(x_i, x_{i+1}), (x_{i+1}, x_{i+2})$ are in non-trivial component. By Corollary 3.4 (x_i, x_{i+2}) is also in a non-trivial component. Now if both (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) are old then we claim that (x_i, x_{i+2}) is also an old pair. Otherwise we have $S_{x_i x_{i+1}} \neq S_{x_i x_{i+2}}$, $S_{x_{i+1} x_{i+2}} \neq S_{x_i x_{i+2}}$ and $S_{x_i x_{i+1}} \neq S_{x_{i+1} x_{i+2}}$, moreover $S_{x_i x_{i+1}} \neq S_{x_{i+2} x_{i+1}}$, $S_{x_{i+1} x_i} \neq S_{x_{i+1} x_{i+2}}$ because there was no circuit at step 2. Now H is a pre-insect with $Z = \emptyset$ and hence $S_{x_i x_{i+2}}$ is not a dictator component. Similarly according to the minimality of the circuit, it is not possible that both (x_i, x_{i+1}) and (x_{i+1}, x_{i+2}) are new. So we may assume that (x_i, x_{i+1}) is old and (x_{i+1}, x_{i+2}) is new. Now again we know that $S_{x_i x_{i+1}} \neq S_{x_{i+1} x_{i+2}}$ and $S_{x_i x_{i+1}} \neq S_{x_i x_{i+2}}$. We note that $S_{x_{i+1} x_{i+2}} \neq S_{x_i x_{i+2}}$ as otherwise we get a shorter circuit. Therefore H is pre-insect with $Z = \emptyset$ and hence by Lemma 7.1 (x_{i+1}, x_{i+2}) is not in a dictator component.

If none of the (x_i, x_{i+1}), (x_{i+1}, x_{i+2}) is in a non-trivial component, then (x_i, x_{i+2}) is implied by the same non-trivial component implying (x_i, x_{i+1}) and hence we get a shorter circuit. So we may assume that (x_i, x_{i+1}) alternate, meaning that if (x_i, x_{i+1}) is implied then (x_{i+1}, x_{i+2}) is in a non-trivial component and vice versa. Now in this case as we argue in the correctness of step (2) there would be an exobiclique in H which is not possible. $\diamond$

The following lemma shows the number of distinct dictator component is at most $2n$.

Lemma 7.3 *The number of distinct dictator non-trivial components is at most $2n$.*

Proof: Note that there are at most n^2 distinct non-trivial components. Consider non-trivial component S_{ab}, S_{ac} such that $S_{ab} \neq S_{ac}$ and $S_{ab} \neq S_{ca}$. It is not difficult to see that S_{bc} is also a non-trivial component as otherwise $S_{ab} = S_{ac}$. Now we must have $S_{bc} = S_{ac}$ or $S_{bc} = S_{ca}$ as otherwise by Lemma 7.1, S_{ab} would not be a dictator non-trivial component. In general if vertex a with vertices $a_1, a_2, ..., a_k$ appear in distinct dictator giant component S_{aa_i}, $1 \leq i \leq k$ then none of the $S_{a_i a_j}$ would be distinct non-trivial component from $S_{aa_1}, S_{aa_2}, ..., S_{aa_k}$. These would imply that there are at most $O(n)$ distinct dictator non-trivial components. $\diamond$

8 Correctness of the Step 6

Theorem 8.1 *By always choosing a sink component in step 5, and taking transitive closure, we cannot create a circuit in D.*

Proof: Suppose by adding a terminal (trivial) component (x, y) into D we create a circuit. Note that none of the $(x, y), (y, x)$ is in D and also (x, y) is not by transitivity on some of the pairs in D as otherwise it would be placed in D. Since (x, y) is a sink pair at the current step of the algorithm, if (x, y) dominates a pair (u, v) in H^+ then (u, v) is in D. The only way that adding (x, y) into D creates a circuit in D is when (x, y) dominates a pair (u, v) while there is a chain $(v, y_1), (y_1, y_2), ..., (y_k, v)$ of pairs in D implying that $(v, u) \in D$. When x, y have the same color $v = y$ and xu is an edge which means (v, u) implies (y, x) and hence $(y, x) \in D$ a contradiction. When x, y have different colors then $u = x$ and yv is an edge and hence $(v, u) \in D$ dominates (y, x) a contradiction. $\diamond$

9 Implementation and complexity

In order to construct digraph H^+, we need to list all the neighbors of each vertex. If x, y in H have different colors then vertex (x, y) of H^+, has d_y out-neighbors where d_y is the degree of y in H. If x, y have the same color then vertex (x, y) has d_x out-neighbors in H^+. For simplicity we assume that $|W| = |B| = n$. For a fixed black vertex x the number of all pairs which are a neighbor of all some vertex (x, z), $z \in V(H)$ is $nd_x + d_{y_1} + d_{y_2} + \ldots + d_{y_n}$, $y_1, y_2, ..., y_n$ are all the white vertices. Therefore it takes $O(nm)$, m is the number of edges in H, to construct H^+. We may use a link list structure to represent H^+. It order to check whether there exists a self-coupled component, it is enough to see whether (a, b) and (b, a) belongs to the same non-trivial component. This can be done in time $O(mn)$. Since we maintain a partial order D once we add a new pair into D we can decide whether we close a circuit or not. Computing $\widehat{D}$ takes $O(n(n + m))$ since there are $O(mn)$ edges in H^+ and there are at most $O(n^2)$ vertices in H^+. Note that the algorithm computes the envelope of D at most twice once at step (3) and once at step (5).

Once a pair (x, y) is added into D, we put an arc from x to y in the partial order and the arc xy gets a time label denoted by $T(x, y)$. $T(x, y)$ is the level in which (x, y) is created. In order to look for a circuit we need to consider a circuit D in which each pair in original. Once a circuit is formed in step (3) by using *Dict* function we can find a dictator component S and store it into set $\mathcal{DT}$. Therefore we spend at most $O(n(m+n))$ time to find all the dictator components. After step (5) we add the rest of the remaining pairs and that takes at most $O(n^2)$. Now it is clear that the running time of the algorithms is $O(n(m + n))$.

10 Constructing a Family of Obstructions

We start with four vertices x_0, x_1, x_2, x_3 such that x_0, x_3 have the same color and opposite to the color of x_1, x_2. There are vertices y_0, y_1, z_1, z_2 such that y_0x_0, x_1y_1, z_1z_2 are independent edges and x_2y_1, x_2z_1, x_2x_0 are edges of H. Each of the x_0, z_1, y_1 are adjacent to each neighbor of x_3. Now consider three independent edges v_1q_1, v_2q_2, v_3q_3 and a new vertex v such that v is adjacent to $v_1, v_2, v_3, x_0, y_1, z_1$. Let pq be an edge independent to x_3z such that qv_1, qv_2, qv_3 are edges of H. Finally we connect z to v_1, v_2, v_3.

Now at step $1 \leq i \neq n - 1$ introduce new vertices u^i, u^i_1, u^i_2 such that u^i, x_3 have the same color and opposite to the color of u^i_1, u^i_2. $u^iu^{i-1}_2, u^iu^i_2$ are edges of H and there are independent

edges $u^iw^i, u_1^iw_1^i, z_1^i, z^i$ such that u_2^i is adjacent to all u^i, w_1^i, z_1^i. Finally $u^nu_2^{n-1}, x_3z, u^1x_2$ are independent edges.

We note that (u^i, x_3) is obtained from $(u^i, u_1^i), (u_1^i, u_2^i), (u_2^i, x_3)$ and (u_2^i, x_3) is implied by (u^{i+1}, x_3) if (u^{i+1}, x_3) is chosen. Therefore (u^1, x_3) is selected when we choose $S_{u^nx_3}$ and hence (x_2, x_3) is implied. We also have the pairs $(x_0, x_1), (x_1, x_2)$ and the pair (x_3, v) is by transitivity on the pairs $(x_3, v_1), (v_1, q_2), (q_2, v)$ implying that the pair (x_3, x_0). According to the Lemma 6.4, $S_{u^nx_3} = S_{x_3q}$ and hence (x_3, q) implies (x_3, v_1). Now we may assume that $(v_1, q_2), (q_2, v)$ are the selected pairs and hence (x_3, v) is selected which implies (x_3, x_0). Therefore we get a circuit $C = (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ if we choose the non-trivial component $S_{u^nx_3}$.

Now we add new vertices in order to construct a dual circuit of C when we choose the component $S_{x_3u^n}$. Consider vertex x_0', x_1' such that x_0', x_3 have the same color and opposite to the color of x_2, x_1'. Let $x_0'y_0', x_1'y_1', z_1'z_2'$ be independent edges such that $u_2^{n-1}x_0', u_2^{n-1}y_1', u_2^{n-1}z_1'$ are edges of H and let x' be a vertex adjacent to $x_0', z_1', y_1', v_1, v_2, v_3$. Now we get a circuit $(v_3, q_2), (q_2, q), (q, x_3), (x_3, v_3)$ where (x_3, v_3) is implied by (x_3, x') and (x_3, x') is by transitivity on the pairs $(x_3, x_0'), (x_0', y_1'), (y_1', x')$.

11 Examples :

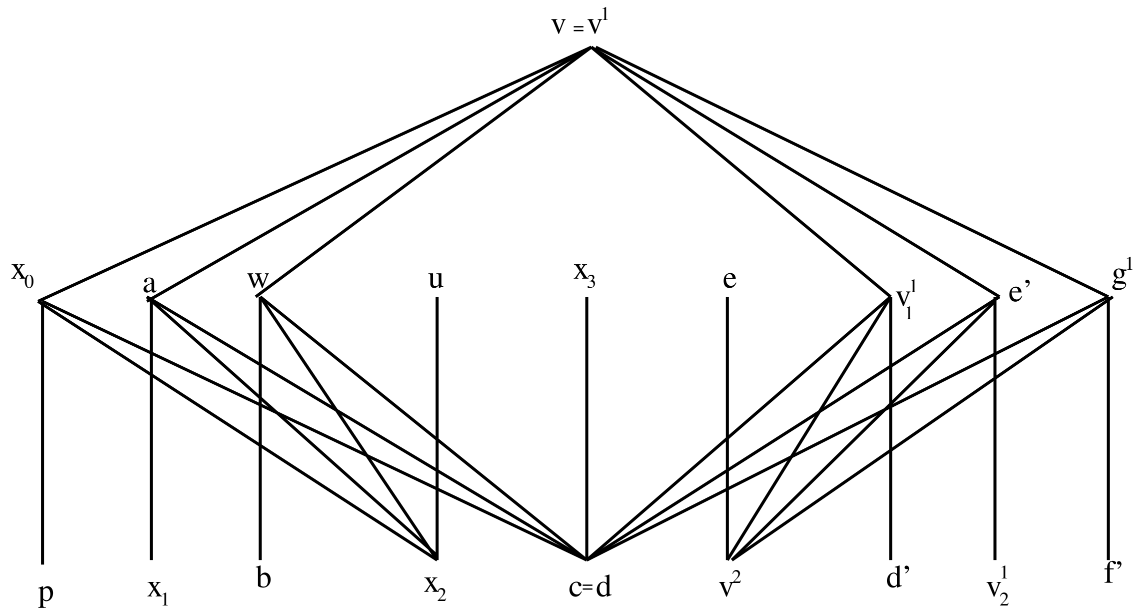

Figure 4: Obstruction

We show that the graph depicted in Figure 4 does not admit the desired ordering. In fact there would be a circuit in both Steps 3 and 4. Suppose we choose non-trivial components $S_{x_0x_1}$ and S_{x_1w} and $S_{x_2x_3}$ and non-trivial components $S_{d'e'}, S_{e'f'}$ at step (2) of the algorithm. We have $(x_1, w) \to (x_1, x_2)$ and $(v_2^1, g^1) \to (v_2^1, v)$ and $(x_3, v) \to (x_3, v_1^1)$. Note that $(x_2, x_3), (x_3, v^2)$ are in the same non-trivial component since x_2, d are adjacent to w while

v^2 is not adjacent to w and d, v^2 are adjacent to v_1^1 while $x_2 v_1^1$ is not an edge of H. All the pairs $(x_0, x_1), (x_1, x_2), (x_3, v_1^1), (v_1^1, v_2^1), (v_2^1, v)$ are placed in D (at step (2) of the algorithm). Now we must add the pairs that are by transitivity and implication closure. In particular $(x_3, v_1^1), (v_1^1, v_2^1), (v_2^1, v)$ imply (x_3, v) and $(x_3, v) \rightarrow (x_3, x_0)$ and hence we have the circuit $(x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_0)$ in D. Note that since d, v, v^2 all are adjacent to v_1^1, e', g^1, choosing any of the $S_{d'e'}, S_{e'd'}$ instead of $S_{d'e'}$ and any of $S_{e'f'}, S_{f'e'}$ instead of $S_{d'e'}$ would yield a circuit in D as long as we choose $S_{x_2x_3}$. Also selecting any two non-trivial components from $S_{x_0x_1}, S_{x_1x_0}, S_{x_1w}, S_{wx_1}$ would also yield a circuit as long as we choose $S_{x_2x_3}$ at step (2). Therefore in order to avoid a circuit at step (3) of the algorithm we must choose $S_{x_3x_2}$. Now if we choose $S_{x_3x_2}$ and choose $S_{d'e'}, S_{e'f'}, S_{x_1x_2}, S_{x_1w}$ at step (2) of the algorithm we also must choose the following pairs :

$(v_2^1, g^1) \rightarrow (v_2^1, v^2)$, $(x_3, x_2) \rightarrow (x_3, x_0)$, (x_0, x_1) and $(x_1, w) \rightarrow (x_1, v)$. Therefore by applying the transitivity we would have (x_0, v) and now $(x_3, x_0), (x_0, v)$ would imply $(x_3, v) \rightarrow (x_3, v_1^1)$. Therefore we have the circuit $(v_1^1, v_2^1), (v_2^1, v^2), (v^2, x_3), (x_3, v_1^1)$. Choosing any two non-trivial components from $S_{d'e'}, S_{e'd'}, S_{e'f'}, S_{f'e'}$ instead of $S_{d'e'}, S_{e'f'}$ would yield a circuit. These imply that H is not an interval bigraph.

12 Conclusion and future work

We mention that a naive approach would yield a simple algorithm with running time $O(n^4)$ as follows. Construct the pair-digraphs H^+ with $O(n^3)$ edges. Briefly speaking there are $O(n^2)$ vertices in H^+ and each vertex of H^+ has at most n neighbors. Therefore the size of H^+ is $O(n^3)$. The algorithm starts with an empty partial order D with n vertices and when a strong component S is chosen the algorithm puts an arc from a to b for $(a, b) \in S$. We run step (2) and if we encounter a circuit (a circuit is a directed cycle in D)at this step then H contains an exobiclique and hence H is not interval bigraph. In step (3) the algorithms computes $\widehat{D}$. Note that computing envelope of set D takes $O(n^3)$ since there are $O(n^3)$ edges in H^+. If there exists a circuit in $\widehat{D}$ then let S be a dictator component for this circuit. In order to detect the circuit we can trace back (by looking at the reverse order of transitive closure and implications) and find a dictator component S involved in creating this circuit. The algorithm repeats step (2) from beginning by fixing $(S')^*$ to be in D and then it performs step (3). At each execution of step (3) one dictator component is detected. If S and S' both are detected as dictator components then H is not an interval bigraphs. By Lemma 7.3 the number of dictator components is at most $2n$. Thus we need to repeat step (2) and (3) at most $2n$ times.

We have introduced an algorithm that works with a pair-digraph in order to produce an ordering for interval bigraph H. Analyzing the behavior of the algorithm when it encounters a circuit gives an insight into structure of forbidden subgraphs of interval bigraphs. We hope our algorithm be a useful tool for obtaining interval bigraph obstructions. As mentioned earlier, several of the ordering problems with forbidden patterns can be transformed to selecting the components of a pair-digraph without creating a circuit.

One of the problems that can be formulated as an ordering without seeing forbidden patterns is a *min ordering (X-underbar) ordering*. A *min ordering* of a digraph H is an ordering of its

vertices $a_1, a_2, \ldots, a_n$, so that the existence of the arcs $a_i a_j, a_{i'} a_{j'}$ with $i < i', j' < j$ implies the existence of the arcs $a_i a_{j'}$. We leave open the following problem.

Problem 12.1 *Is there a polynomial time algorithm that decides whether an input digraph H admits a min ordering?*

As mentioned, interval bigraphs and interval digraphs became of interest in new areas such as graph homomorphisms. The digraphs admitting min ordering are closely related to interval digraphs and they are useful in research area such as graph homomorphisms and constraint satisfaction problems.

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